ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 3, Summer 1971

AN EXAMPLE CONCERNING ALGEBRAS OF MEASURABLE FUNCTIONS[1]

ANTHONY W. HAGER

If $\mathcal{A}$ is a point-separating σ-algebra of subsets of the set S, a function $f:S \to R$ ($R =$ the reals) is called $\mathcal{A}$-measurable if $f^{\leftarrow}(I) \in \mathcal{A}$ for each interval I. The family of such functions can be equipped with pointwise addition, multiplication, and multiplication by real numbers, and becomes what we shall call an algebra of measurable functions on the set S. Such an algebra A has many properties, among them: A is closed under uniform convergence on S, and is regular in the sense of von Neumann (i.e., given f there is g with $f^2g = f$). In fact, these two properties characterize such algebras. That is: *Let A be a point-separating subalgebra of* R^S, *with* $1 \in A$. *If A is regular and closed under uniform convergence, then A is an algebra of measurable functions on* S. (This observation seems essentially due to B. Brainerd. This is discussed, and other similar results surveyed, in [4].) In the present context, the proof of this is not hard; we sketch it: since A is closed under uniform convergence, it is a lattice and the family $\mathcal{A} \equiv \{\operatorname{coz}(f) : f \in A\}$ is closed under countable union (where $\operatorname{coz}(f) = \{s \in S : f(s) \neq 0\}$); since A is regular, $\mathcal{A}$ is closed under complementation, and is an σ-algebra; easy approximation techniques now show that A coincides with the algebra of $\mathcal{A}$-measurable functions on S.

This note concerns the question of whether or not a certain weakening of the hypotheses in the above theorem entails the conclusion. We formulate the question in the following paragraphs. To avoid repetition, from now on each A is to be a point-separating subalgebra of some R^S, with $1 \in A$.

Evidently, if A is regular, then A is closed under inversion (of functions without zeros). And, if A is regular and closed under uniform convergence, then A induces a (completely regular Hausdorff) "P-space" topology on S, i.e., one in which whenever a G_δ-set contains a point, it is a neighborhood of the point. This can be proved directly, but it is seen more easily if we assume (*via* the above theorem) that A is the algebra of $\mathcal{A}$-measurable functions; then $\mathcal{A}$ is an open basis which is closed under countable intersection.

Received by the editors July 25, 1969.

AMS 1970 *subject classifications.* Primary 06A70, 28A20, 54C30; Secondary 06A40, 46A30, 46E25, 46E30, 54C40.

[1]This research was partially supported by a grant from Wesleyan University, and by the National Science Foundation of the United States under grant GP-18825.