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SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS WITH SMALL KERNELS

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ABSTRACT. In 1975, Grimmer and Seifert studied a linear integro-differential equation with weakly singular kernel, $C(t,s)$, by means of a Razumikhin technique. They obtained bounded solutions from bounded forcing functions. Their conditions centered on small integrals of the kernel with respect to the second coordinate, s. On the last page of their paper they express the desire to obtain L^p solutions from L^p forcing functions. A recent result for singular integral equations makes it possible to answer the question. Here, we study a variety of integro-differential equations with singular kernels including linear, nonlinear, scalar, vector and resolvent equations by means of Liapunov functionals. We do obtain the types of L^p solutions from L^p perturbations. The point here is that there is a loose principle of the following type. Generally, but not always, Razumikhin techniques integrate the second coordinate and obtain bounded solutions, while Liapunov functionals integrate the first coordinate of the kernel and obtain L^p solutions. For decades, investigators have discussed and debated which technique was the "best." In fact, neither is best. They perform different sets of tasks, with a non-empty intersection.

1. Introduction. We study a scalar integro-differential equation of the form

$$x'(t) = f(t) - h(t, x(t)) - \int_0^t C(t,s) q(s, x(s))\, ds, \tag{1}$$

and also a linear vector equation, together with its resolvent. The objective is to determine qualitative properties of solutions when

$$\textit{there exists a } p \in [1, \infty) \textit{ with } f \in L^p[0, \infty), \tag{2}$$

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