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THE COMPLEXITY OF FREDHOLM EQUATIONS OF THE SECOND KIND: NOISY INFORMATION ABOUT EVERYTHING

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ABSTRACT. We study the complexity of Fredholm problems of the second kind $u - \int_\Omega k(\cdot, y)u(y)\,dy = f$. Previous work on the complexity of this problem has assumed that Ω was the unit cube I^d. In this paper, we allow Ω to be part of the data specifying an instance of the problem, along with k and f. More precisely, we assume that Ω is the diffeomorphic image of the unit d-cube under a C^{r_1} mapping $\rho: I^d \to I^l$. In addition, we assume that $k \in C^{r_2}(I^{2l})$ and $f \in C^{r_3}(I^l)$. Using a change of variables, we can reduce this problem to an integral equation over I^d. Our information about the problem data consists of function evaluations, contaminated by δ-bounded noise. Error is measured by the max norm. We show that the problem is unsolvable if $r_1 = 1$ and $d < l$. Hence we assume that either $r_1 \geq 2$ or $d = l$ in what follows. We find that the nth minimal error is bounded from below by $\Theta(n^{-\mu_1} + \delta)$ and from above by $\Theta(n^{-\mu_2} + \delta)$, where

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\} \quad \text{and} \quad \mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

The upper bound is attained by a noisy modified Galerkin method, which can be efficiently implemented by a two-grid algorithm. We thus find bounds on the ε-complexity of the problem, these bounds depending on the cost $\mathbf{c}(\delta)$ of calculating a δ-noisy function value. As an example, if $\mathbf{c}(\delta) = \delta^{-b}$, we find that the ε-complexity is between $(1/\varepsilon)^{b+1/\mu_1}$ and $(1/\varepsilon)^{b+1/\mu_2}$.

1. Introduction.

We are interested in the worst case complexity of solving Fredholm problems of the second kind

$$u(s) - \int_\Omega k(s,t)u(t)\,dt = f(s) \qquad \forall\, s \in \Omega. \tag{1.1}$$

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