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ON A GENERALIZED INTEGRAL EQUATION WHICH ORIGINATES FROM A PROBLEM IN DIFFUSION THEORY

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1. Introduction. Let $B_x(s)$ be a reflecting Brownian motion on $(0,\infty)$ with $B_x(0)=x$. Let τ^+ be the first time, s, that the sojourn time of $(1,\infty)$, for B_x up to time s, exceeds the sojourn time of $[0,1]$ up to time s, and define $Y^+ = B_x(\tau^+)$. In [**2**] it was established that, for $0<x<1$, the probability density of Y^+ is $\Pi(x,y)$ in the sense that $\mathbf{P}^x(Y^+ \in (1+y, 1+y+dy)) = \Pi(x,y)\,dy$, where Π satisfies

$$\int_0^\infty (\cosh\theta\cos\theta y + \sinh\theta\sin\theta y)\Pi(x,y)\,dy = \cosh\theta x, \qquad \theta>0. \tag{1}$$

For further discussion of this remarkable result and additional references see [**1**].

In [**2**] the closed form solution to the above problem is obtained by ad hoc methods in the form .

$$\Pi(x,y) = \frac{\cosh\frac{1}{2}\pi y(\sinh\frac{1}{2}\pi y\cos\frac{1}{2}\pi x)^{\frac{1}{2}}}{\sqrt{2}(\sinh^2\frac{1}{2}\pi y + \cos^2\frac{1}{2}\pi x)}. \tag{2}$$

A constructive proof of this result was given in [**3**] by using Laplace transform methods to show that the associated integral equation

$$\int_0^\infty \Big(\sin\Big(\frac{\pi}{4}+\theta\Big)e^{-\theta y} + \sin\Big(\frac{\pi}{4}-\theta\Big)e^{\theta y}\Big)\Pi(x,y)\,dy = \sqrt{2}\cosh\theta x, \tag{3}$$

where x and θ are complex, admits a solution $\Pi(x,y)$ in convolution form, namely,

$$\Pi(x,y) = \sqrt{(2\pi)}\int_0^y \frac{G(x,y-\nu)\,d\nu}{\sqrt{(\sinh\frac{1}{2}\pi\nu)}}, \tag{4}$$

where

(5)

$$G(x,\tau) = G(x,-\tau) = \frac{\sqrt{\pi}}{16}\Big\{\Big[\cosh\frac{1}{2}\pi(x+\tau)\Big]^{-3/2} + \Big[\cosh\frac{1}{2}\pi(x-\tau)\Big]^{-3/2}\Big\}.$$