JOURNAL OF COMMUTATIVE ALGEBRA
Volume 8, Number 2, Summer 2016

REDUCED SYMMETRIC ALGEBRAS AND LINEAR SYZYGIES

MARK R. JOHNSON

1. Introduction. Let M be a finitely generated module over a Noetherian domain A. A basic condition one would often like to know is the integrality of the symmetric algebra $S_A(M)$. A fundamental obstruction for this to hold is a restriction on the (local) number of generators: $\mu(M_p) \leq \dim A_p + \operatorname{rank} M - 1$ for every non-minimal prime ideal p of A. This condition is known as the $\mathcal{F}_1$ condition.

Unfortunately, the $\mathcal{F}_1$ condition alone is usually not enough to guarantee integrality; however, if the symmetric algebra happens to be equidimensional, then one may conclude (at least if A has finite Krull dimension) that its associated spectrum is irreducible [**9**]. Thus, the integrality would follow if one also knew the reducedness of $S_A(M)$. It turns out that there are other situations where reducedness ensures the integrality.

Recently, Simis, Ulrich and Vasconcelos [**7**] have discovered one such case for the Zariski tangent algebra, which is the symmetric algebra $S_A(\Omega_{A/k})$ of the module of differentials of an algebra A essentially of finite type over a field k. For example, if A is a local isolated singularity over the perfect field k, then the reducedness of the Zariski tangent algebra of A implies its integrality–as long as A is not defined by "too many" quadrics. They also establish a similar result for its normality, showing that it must in such a case coincide with its reflexive closure.

In this note, these results are extended to arbitrary modules, giving similar criteria for a reduced symmetric algebra to be integral, or a normal symmetric algebra to be its reflexive closure. The conditions translate into the module having sufficiently few "linear" relations. This also serves to clarify, and somewhat simplify, this part of the work of [**7**].

2010 AMS *Mathematics subject classification.* Primary 13A30.

Keywords and phrases. Symmetric algebra, linear syzygy, linear type.

Received by the editors on January 4, 2013, and in revised form on August 19, 2013.

DOI:10.1216/JCA-2016-8-2-227