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A UNIVERSAL COEFFICIENT THEOREM FOR GAUSS'S LEMMA

WILLIAM MESSING AND VICTOR REINER

To Jürgen Herzog on his 70th birthday

ABSTRACT. We shall prove a version of Gauß's lemma. It works in $\mathbf{Z}[\mathbf{a},\mathbf{A},\mathbf{b},\mathbf{B}]$ where $\mathbf{a} = \{a_i\}_{i=0}^m$, $\mathbf{A} = \{A_i\}_{i=0}^m$, $\mathbf{b} = \{b_i\}_{j=0}^n$, $\mathbf{B} = \{B_j\}_{j=0}^n$, and constructs polynomials $\{c_k\}_{k=0,\dots,m+n}$ of degree at most $\binom{m+n}{n}$ in each variable set $\mathbf{a},\mathbf{A},\mathbf{b},\mathbf{B}$, with this property: setting

$$\sum_k C_k X^k = \sum_i A_i X^i \cdot \sum_j B_j X^j;$$

for elements a_i, A_i, b_j, B_j in any commutative ring R satisfying

$$1 = \sum_i a_i A_i = \sum_j b_j B_j,$$

the elements $c_k = c_k(a_i, A_i, b_j, B_j)$ satisfy $1 = \sum_k c_k C_k$.

1. The statement. Let R be a commutative ring. Consider two elements $A(X) = \sum_{i=0}^m A_i X^i$ and $B(X) = \sum_{j=0}^n B_j X^j$ in $R[X]$, with the product $C(X) = A(X)B(X) = \sum_{k=0}^{m+n} C_k X^k$, so that one has $C_k = \sum_{i+j=k} A_i B_j$. A version of Gauß's lemma, called *Gauß-Joyal de pauvre* in [**6**, Section II, Lemma 2.6], asserts the following.

Proposition 1. *If both $A(X), B(X)$ have the property that their coefficient sequences generate the unit ideal R, then the same is true of their product $C(X)$, that is, $(A_0, \dots, A_m) = R = (B_0, \dots, B_n)$ implies $(C_0, \dots, C_{m+n}) = R$.*

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