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ON THE REGULARITY OF CONFIGURATIONS OF $\mathbf{F}_q$-RATIONAL POINTS IN PROJECTIVE SPACE

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Dedicated to Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. We are interested in the smallest number $s = s(n, q)$ such that, for any given n distinct $\mathbf{F}_q$-rational points $P_1, \ldots, P_n \in \mathbf{P}^{n-1}$, there exists a hypersurface H of degree s and defined over $\mathbf{F}_q$ such that $P_1, \ldots, P_{n-1} \in H, P_n \notin H$. Alternately, $s(n, q)$ is the maximal Castelnuovo-Mumford regularity of a set of n $\mathbf{F}_q$-rational points in some projective space. Finally, $s(n, q)$ is the index of stability of certain one-dimensional local Cohen-Macaulay rings.

1. Introduction.

Let K be a field and $\mathbf{P}^k(K)$ the set of all K-rational points in the k-dimensional projective space over K $(k \geq 1)$. We consider subsets $\mathfrak{X} \subset \mathbf{P}^k(K)$ with $\deg \mathfrak{X} = |\mathfrak{X}| =: n \geq 1$.

Let $R = K[X_0, \ldots, X_k]$ be the polynomial ring over K in the variables $X_0, \ldots, X_k$, and let

$$I_{\mathfrak{X}} := (\{F \in R \text{ homogenous} \mid F(P) = 0 \text{ for all } P \in \mathfrak{X}\})$$

be the homogenous vanishing ideal of $\mathfrak{X}$. Then $S := R/I_{\mathfrak{X}}$ is a standard graded ring. The Hilbert function $H_{\mathfrak{X}}$ of $\mathfrak{X}$ is defined as

$$H_{\mathfrak{X}}(d) = \dim_K S_d \quad (d \in \mathbf{N}),$$

where S_d is the homogenous component of degree d of S. As is well known, there is a number $r_{\mathfrak{X}}$, such that $H_{\mathfrak{X}}(d) = n$ for $d \geq r_{\mathfrak{X}}$ and $H_{\mathfrak{X}}(r_{\mathfrak{X}} - 1) < n$. It is called the *regularity* (Castelnuovo-Mumford regularity) of $\mathfrak{X}$. For $0 \leq d \leq r_{\mathfrak{X}}$, the function $H_{\mathfrak{X}}$ is strictly increasing; hence, $r_{\mathfrak{X}} \leq n - 1$.

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