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THE EVENTUAL STABILITY OF DEPTH, ASSOCIATED PRIMES AND COHOMOLOGY OF A GRADED MODULE

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1. Introduction. The asymptotic stability of several homological invariants of graded pieces of a graded module has attracted quite a lot of attention over the last decades. An early important result was the proof by Brodmann of the eventual stabilization of associated primes of the powers of an ideal in a Noetherian ring ([**1**]).

We provide in this text several stability results together with estimates of the degree from which it stabilizes. One of our initial goals was to obtain a simple proof of the tameness result of Brodmann in [**2**] for graded components of cohomology over rings of dimension at most two. This is achieved in the last section and gives a slight generalization of what is known, as our result (Theorem 7.4) applies to Noetherian rings of dimension at most two that are either local or the epimorphic image of a Gorenstein ring. Recall that Cutkosky and Herzog provided examples in [**3**] showing that tameness does not hold over rings of dimension three (even over such nice local rings).

Besides this result, we establish, for a graded module M over a polynomial ring S (in finitely many variables, with its standard grading) over a commutative ring R, stability results for the depth and cohomological dimension of graded pieces with respect to a finitely generated R-ideal I. It follows from our results that the cohomological dimension of M_μ with respect to I is constant for $\mu > \operatorname{reg}(M)$, and the depth with respect to I is at least equal to its eventual value for $\mu > \operatorname{reg}(M)$ and stabilizes when it reaches this value for some M_μ with $\mu > \operatorname{reg}(M)$. See Propositions 3.1 and 4.9 for more precise results.

Recall that $\operatorname{reg}(M) \in \mathbf{Z}$ when $M \neq 0$ is finitely generated and R is Noetherian.

When R is Noetherian, $\mathfrak{p} \in \operatorname{Spec}(R)$ is associated to M_μ for some μ if and only if $\mathfrak{p} = \mathfrak{P} \cap R$ for $\mathfrak{P}$ associated to M in S, and the sets of

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