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ON PRIME MODULES AND DENSE SUBMODULES

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ABSTRACT. Let R be a commutative ring with identity, and let M be a unital R-module. A submodule N of M is called a dense submodule, if $M = \sum_{\varphi} \varphi(N)$ where φ runs over all the R-morphisms from N into M. An R-module M is called a π-module if every nonzero submodule is dense in M. This paper makes some observations concerning prime modules and π-modules over a commutative ring. It is shown that an R-module M is a prime module if and only if every nonzero cyclic submodule of M is a dense submodule of M. Moreover, for modules with nonzero socles and co-semisimple modules over any ring and for all finitely generated modules over a principal ideal domain (PID), the two concepts π and prime are equivalent. Rings R, over which the two concepts π and prime are equivalent for all R-modules, are characterized. Also, it is shown that, if M is a π-module over a domain R with $\dim(R) = 1$, then either M is a homogeneous semisimple module or a torsion free module. In particular, if M is a multiplication module over a domain R with $\dim(R) = 1$, then M is a π-module if and only if either M is a simple module or R is a Dedekind domain and M is a faithful R-module.

0. Introduction. All rings in this article are commutative with identity and modules are unital. For a ring R we denote by $\dim(R)$ the classical Krull dimension of R and for a module M we denote by $\operatorname{soc}(M)$ and $\operatorname{Ann}(M)$ the socle and the annihilator of M, respectively.

Let R be a ring. We recall that a nonzero R-module M is said to be a *prime module* if $\operatorname{Ann}(N) = \operatorname{Ann}(M)$ for each non-zero submodule N of M, i.e., $rx = 0$ for $x \in M$, $r \in R$ implies that $x = 0$ or $rM = (0)$. We call a proper submodule N of an R-module M a *prime submodule* of M if M/N is a prime module, i.e., whenever $rm \in N$, then either $m \in N$ or $rM \subseteq N$ for any $r \in R$, $m \in M$. Thus, N is a prime submodule

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