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GENERALIZING SPERNER'S LEMMA TO A FREE MODULE OVER A SPECIAL PRINCIPAL IDEAL RING

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ABSTRACT. Sperner's lemma states that if $\mathcal{A}$ is an anti-chain from the power set of an n-element set, then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Rota and Harper provide the following q-analogue to a number of classical generalizations of Sperner's lemma: *if $\mathcal{A}$ is an l-chain-free family of subspaces of a finite vector space $\mathbf{F}_q^n$, then $\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\dim(A)}_q} \leq l$ and $|\mathcal{A}|$ is bounded by the sum of the l largest Gaussian coefficients $\binom{n}{k}_q$.* In this work, the original Sperner's lemma as well as Rota and Harper's result are extended to multiple generalizations in the setting of a finitely-generated free module over a finite special principal ideal ring.

1. Introduction. As given in [**10, 11**], Sperner's lemma (regarding finite sets) states that the cardinality of any anti-chain (i.e., collection of incomparable subsets) from the power set $\mathcal{P}([n])$ of an n-element set $[n]$ does not exceed the combination $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Some well-known n-set generalizations of Sperner's lemma are stated in the next theorem. (As in [**1**], we use the convention that an *l-chain free family* is a collection of subsets that contain no chain, $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l$, of length l.)

Theorem 1.1 ([**2, 6–8, 12**]). *Let $\mathcal{A}$ be an l-chain-free family of subsets from $\mathcal{P}([n])$. Then*: (a) $\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq l$.

(b) $|\mathcal{A}|$ *is bounded by the sum of the l largest values* $\binom{n}{k}$, $0 \leq k \leq n$.

(c) *Letting $\mathcal{S}_k$ denote the set of all k-element subsets in $\mathcal{P}([n])$, there is equality in* (a) *and* (b) *when $\mathcal{A}$ consists of the l largest sets $\mathcal{S}_k$, $0 \leq k \leq n$.*

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