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ARITHMETICAL RINGS SATISFY THE RADICAL FORMULA

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ABSTRACT. In this paper we prove that every arithmetical ring satisfies the radical formula.

1. Introduction. Throughout this article, rings are assumed to be commutative with unity and modules are assumed to be unitary. Let R be a ring and M an R-module. A proper submodule N of M is said to be a prime submodule of M if $ax \in N$ for $a \in R$ and $x \in M$ implies that either $aM \subseteq N$ or $x \in N$. In this case, $P = (N : M)$ is a prime ideal of R and N is said to be a P-prime submodule of M.

Let N be a proper submodule of M. The intersection of all prime submodules of M containing N is denoted by $\mathrm{rad}\,(N)$. If no prime submodule of M exists containing N, then $\mathrm{rad}\,(N)$ is defined to be M.

Also, for any subset N of M, the envelope of N, $E(N)$ is defined to be:

$$E(N) = \{x \mid x = ay, a^n y \in N, \text{ for some } a \in R, y \in M \text{ and } n \in \mathbf{N}\}.$$

In general $E(N)$ is not a submodule of M. It is clear that $\langle E(N)\rangle$, the submodule generated by $E(N)$, is contained in $\mathrm{rad}\,(N)$. M is said to satisfy the radical formula (M s.t.r.f.), if for every submodule N of M, $\langle E(N)\rangle = \mathrm{rad}\,(N)$. Furthermore, if every R-module satisfies the radical formula, then R is said to satisfy the radical formula.

A ring R is said to be an arithmetical ring if, for all ideals I, J and K of R, we have $I + (J \cap K) = (I + J) \cap (I + K)$. Obviously Prüfer domains and, in particular, Dedekind domains are arithmetical.

The question of what type of rings s.t.r.f. was considered in [**1, 4, 6–9**]. In [**1**], it was shown that every arithmetical ring with $\dim\,(R) \leq 1$

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