JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 2, Summer 2012

SEPARATORS OF ARITHMETICALLY COHEN-MACAULAY FAT POINTS IN $\mathbf{P}^1 \times \mathbf{P}^1$

ELENA GUARDO AND ADAM VAN TUYL

ABSTRACT. Let $Z \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ be a set of fat points that is also arithmetically Cohen-Macaulay (ACM). We describe how to compute the degree of a separator of a fat point of multiplicity m for each point in the support of Z using only a numerical description of Z. Our formula extends the case of reduced points which was previously known.

1. Introduction. Fix an algebraically closed field k of characteristic zero. Let $R = k[x_0, x_1, y_0, y_1]$ be the $\mathbf{N}^2$-graded polynomial ring with $\deg x_i = (1,0)$ for $i = 0,1$ and $\deg y_i = (0,1)$ for $i = 0,1$. The ring R is the coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$. Consider now a set of points $X = \{P_1, \ldots, P_s\} \subset \mathbf{P}^1 \times \mathbf{P}^1$, and fix positive integers $m_1, \ldots, m_s$. The goal of this note is to study some of the properties of the scheme $Z = m_1P_1 + \cdots + m_sP_s$ of fat points (precise definitions are deferred until the next section). In particular, we are interested in describing the separator of P_i of multiplicity m_i.

Recall that for sets of points $X = \{P_1, \ldots, P_s\} \subseteq \mathbf{P}^n$, a homogeneous form $F \in k[\mathbf{P}^n]$ is called a *separator* of $P \in X$ if $F(P) \neq 0$, but $F(Q) = 0$ for all $Q \in X \setminus \{P\}$. Over the years, a number of authors have shown how to exploit information about the separator of a point to describe properties of the set of reduced points $X \subseteq \mathbf{P}^n$ (e.g., see [**1–4, 11, 13, 14**]). In a series of papers, the authors, along with Marino, (see [**6, 10**]) generalized some of these results by studying separators of fat points, a family of non-reduced points. Roughly speaking, a *separator of a point P_i of multiplicity m_i* and the *degree of a point P_i of multiplicity m_i* are defined in terms of the generators of $I_{Z'}/I_Z$ in R/I_Z where $I_{Z'}$ is the defining ideal of $Z' = m_1P_1 + \cdots + (m_i - 1)P_i + \cdots + m_sP_s$.

2010 AMS *Mathematics subject classification.* Primary 13D40, 13D02, 14M05.
Keywords and phrases. Separators, fat points, Cohen-Macaulay, Hilbert function.
The second author acknowledges the support of NSERC.
Received by the editors on May 28, 2010, and in revised form on June 28, 2011.
DOI:10.1216/JCA-2012-4-2-255