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COMULTIPLICATION MODULES OVER COMMUTATIVE RINGS II

YOUSEF AL-SHANIAFI AND PATRICK F. SMITH

ABSTRACT. Let R be a commutative ring with identity. A unital R-module M is a comultiplication module provided that, for each submodule N of M, there exists an ideal A of R such that N is the set of elements m in M such that $Am = 0$. It is proved that every comultiplication module with zero radical is semisimple. Moreover, for any comultiplication module M, every submodule has a unique complement and a unique closure in M. Every Noetherian comultiplication module is an Artinian quasi-injective module. In case R is a semilocal ring containing precisely n distinct maximal ideals, for some positive integer n, every comultiplication R-module has Goldie dimension at most n. On the other hand, if R is a ring with finite Goldie dimension n, for some positive integer n, then it is proved that certain faithful comultiplication R-modules have hollow dimension at most n.

1. Introduction. This paper is a continuation of [**1**]. Throughout R is a ring with identity and M is a unitary right R-module. Moreover, unless stated otherwise, *R will always denote a commutative ring.* Given submodules N and L of M, we denote by $(N :_R L)$ the set of elements r in R such that $rL \subseteq N$. Note that $(N :_R L)$ is the annihilator in R of the R-module $(L + N)/N$ and is an ideal of R. In particular, if N is a submodule of M and $m \in M$, then $(N :_R Rm)$ will be denoted simply by $(N :_R m)$, so that $(N :_R m) = \{r \in R : rm \in N\}$. On the other hand, if N is again a submodule of M and A is an ideal of R, then $(N :_M A)$ is the set of elements m in M such that $Am \subseteq N$, and it is clear that $(N :_M A)$ is a submodule of M. Recall that M is a *comultiplication module* if, for each submodule N of M, there exists an ideal A of R such that $N = (0 :_M A)$. The first result is taken from [**2**, Theorem 3.17 (d)].

Lemma 1.1. *Every submodule of a comultiplication module is also a comultiplication module.*

Proof. Clear. ◻

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