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MULTIPLICATIVE INVARIANTS AND LENGTH FUNCTIONS OVER VALUATION DOMAINS

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ABSTRACT. The notion of length function ℓ of $\operatorname{Mod} R$ was introduced by Northcott and Reufel in [**6**]. They described length functions when R is a valuation domain. Vámos [**11**] investigated additive functions for Noetherian rings. These functions take values that are either nonnegative real numbers or ∞. We define a multiplicative invariant as a map μ from $\operatorname{Fin} R$, the class of finitely generated R-modules, to a partially ordered multiplicative semigroup Γ, such that $\mu(X) = \mu(Y)\mu(X/Y)$, for $Y \subseteq X$ finitely generated. We investigate the annihilator sets of finitely generated modules over valuation domains. The results we find allow us to show that a certain map $\mu_{\mathcal{I}}$ is a multiplicative invariant that enjoys a universal property. Using $\mu_{\mathcal{I}}$ we re-obtain a description of length functions over valuation domains, in an alternative way to that in [**6**].

Introduction. The starting point of the present paper is the notion of length function of $\operatorname{Mod} R$, introduced by Northcott and Reufel, in the 1965 paper [**6**], as a generalization of the classical Jordan-Hölder length of modules. Namely, if R is any ring, a real-valued map ℓ defined on $\operatorname{Mod} R$ is a length function if for any left R-modules $N \subseteq M$, we have $\ell(M) = \ell(N) + \ell(M/N)$, and $\ell(M) = \sup\{\ell(X)\}$, where X ranges over the finitely generated submodules of M. Northcott and Reufel gave some general results and characterized length functions over valuation domains. Shortly after, Vámos in [**11**] distinguished the two defining properties of a length function, calling a real-valued function ℓ additive if it satisfies the first above property, and upper continuous if it satisfies the second one. In fact, it is easy to show that these properties are independent (see, for instance, our next Remark 3.1). Vámos thoroughly investigated additive functions over Noetherian rings.

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