SOME VARIANTS OF MACAULAY’S
AND MAX NOETHER’S THEOREMS

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Dedicated to Ralf Fröberg on the occasion of his 65th birthday

ABSTRACT. We use residue currents on toric varieties to obtain bounds on the support of solutions to polynomial ideal membership problems. Our bounds depend on the Newton polytopes of the polynomial systems and are therefore well adjusted to sparse systems of polynomials. We present variants of classical results due to Macaulay and Max Noether.

1. Introduction. Let $F_1, \ldots, F_m$, and $\Phi$ be polynomials in $\mathbb{C}^n$. Assume that $\Phi$ vanishes on the common zero set of the $F_j$. Then Hilbert’s Nullstellensatz asserts that there are polynomials $G_1, \ldots, G_m$ such that

\[(1.1) \quad \sum_{j=1}^{m} F_j G_j = \Phi^\nu\]

for some integer $\nu$ large enough. The following bound of the degrees of the $F_j$ and $\nu$ was obtained by Kollár, [19], for $d \neq 2$, and by Jelonek, [18], for $d = 2$ and $m \leq n$:

Assume that $\deg F_j \leq d$. Then one can find $G_j$ so that (1.1) holds for some $\nu \leq d_{\min}(m,n)$ and

\[(1.2) \quad \deg (F_j G_j) \leq (1 + \deg \Phi)d_{\min}(m,n).\]

For $d = 2$ and $m \geq n + 1$ the best bound is due to Sombra [26]: the factor $d_{\min}(m,n)$ in (1.2) should then be replaced by $2^{n+1}$. Kollár’s and Jelonek’s bounds are sharp; the original formulations also take into account different degrees of the $F_j$. In many cases, however, one can do much better. Classical results due to Max Noether, [23], and

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