JOURNAL OF COMMUTATIVE ALGEBRA
Volume 2, Number 3, Fall 2010

STANLEY DECOMPOSITIONS AND HILBERT DEPTH IN THE KOSZUL COMPLEX

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ABSTRACT. Stanley decompositions of multigraded modules M over polynomial rings have been discussed intensively in recent years. There is a natural notion of depth that goes with a Stanley decomposition, called the *Stanley depth*. Stanley conjectured that the Stanley depth of a module M is always at least the (classical) depth of M. In this paper we introduce a weaker type of decomposition, which we call *Hilbert decomposition*, since it only depends on the Hilbert function of M, and an analogous notion of depth, called *Hilbert depth*. Since Stanley decompositions are Hilbert decompositions, the latter set upper bounds to the existence of Stanley decompositions. The advantage of Hilbert decompositions is that they are easier to find. We test our new notion on the syzygy modules of the residue class field of $K[X_1, \dots, X_n]$ (as usual identified with K). Writing $M(n,k)$ for the k-th syzygy module, we show that the Hilbert depth of $M(n,1)$ is $\lfloor (n+1)/2 \rfloor$. Furthermore, we show that, for $n > k \geq \lfloor n/2 \rfloor$, the Hilbert depth of $M(n,k)$ is equal to $n-1$. We conjecture that the same holds for the Stanley depth. For the range $n/2 > k > 1$, it seems impossible to come up with a compact formula for the Hilbert depth. Instead, we provide very precise asymptotic results as n becomes large.

1. Introduction. In recent years *Stanley decompositions* of multigraded modules over polynomial rings $R = K[X_1, \dots, X_n]$ have been discussed intensively. Such decompositions, introduced by Stanley in [**14**], break the module M into a direct sum of graded vector subspaces, each of which is of type Sx where x is a homogeneous element and $S = K[X_{i_1}, \dots, X_{i_d}]$ is a polynomial subalgebra. Stanley conjectured that one can always find such a decomposition in which $d \geq \operatorname{depth} M$ for each summand. (For unexplained terminology of commutative algebra we refer the reader to [**3**].)

Research partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory."
Received by the editors on September 7, 2009, and in revised form on March 23, 2010.
DOI:10.1216/JCA-2010-2-3-327