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COMMUTING NILPOTENT MATRICES AND ARTINIAN ALGEBRAS

ROBERTA BASILI, ANTHONY IARROBINO AND LEILA KHATAMI

ABSTRACT. Fix an $n \times n$ nilpotent matrix B whose Jordan blocks are given by the partition P of n. Consider the ring $\mathcal{C}_B \subset \mathrm{Mat}_n(\mathsf{k})$ of $n \times n$ matrices with entries in an algebraically closed field k that commute with B, and its subset, the variety $\mathcal{N}_B \subset \mathcal{C}_B$ of those that are nilpotent. Then $\mathcal{N}_B$ is an irreducible algebraic variety: so there is a Jordan block partition $Q(P)$ of the generic matrix $A \in \mathcal{N}_B$, that is greater than any other Jordan partition occurring for elements of $\mathcal{N}_B$. What is $Q(P)$? We here introduce an algebra $\mathcal{E}_B$ whose radical is $\mathcal{U}_B$, a maximal nilpotent subalgebra of $\mathcal{N}_B$. We study the poset $\mathcal{D}_P$, related to the digraph used by Oblak and Košir [**13**]. Using our results, we give new, simpler proofs for much of what is known about $Q(P)$, often clarifying or reducing the assumptions needed.

1. Introduction. We denote by k an algebraically closed field, and by $R = \mathsf{k}\{x, y\}$ the completed regular local ring–the power series ring–in two variables. We fix an n-dimensional k-vector space V, and consider the ring $\mathrm{End}_{\mathsf{k}}(V)$ of k-endomorphisms of V. We denote by $P \to n$ a partition $P = (p_1, \dots, p_t), p_1 \geq \cdots \geq p_t$ of $n = |P| = \sum p_i$. We will use the alternate notation $P = \{n_i\} = (n_{p_1}, \dots, n_1)$ where $n_i = \#$ parts equal to i, $t = \sum n_i$ and $n = \sum i n_i$. The notation $P = (4^2, 2^3)$ denotes $n_4 = 2, n_2 = 3$, so $P = (4, 4, 2, 2, 2)$. We denote by $S_P = \{i \mid n_i > 0\}$. We denote by J_P a nilpotent endomorphism of V whose Jordan decomposition has blocks given by the partition P: we will fix later a particular basis $\mathcal{V}$ of V, in which J_P has Jordan block matrix B. We denote by $\mathcal{C}_B$ the centralizer of B in the matrix ring $\mathrm{Mat}_n(\mathsf{k})$, and by $\mathfrak{J}_B$ the Jacobson radical of $\mathcal{C}_B$. We denote by $\mathcal{N}_B$ the

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