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SHORT KOSZUL MODULES

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To Ralf Fröberg, on his 65th birthday

ABSTRACT. This article is concerned with graded modules M with linear resolutions over a standard graded algebra R. It is proved that if such an M has Hilbert series $H_M(s)$ of the form $ps^d + qs^{d+1}$, then the algebra R is Koszul; if, in addition, M has constant Betti numbers, then $H_R(s) = 1+es+(e-1)s^2$. When $H_R(s) = 1+es+rs^2$ with $r \leq e-1$, and R is Gorenstein or $e = r+1 \leq 3$, it is proved that generic R-modules with $q \leq (e-1)p$ are linear.

Introduction. We study homological properties of graded modules over a standard graded commutative algebra R over a field k; recall that this means that R_0 equals k and R is generated over k by finitely many elements of degree one.

Unless R is a polynomial ring, any general statement about R-modules necessarily concerns modules of infinite projective dimension. Various attractive conjectures have been based on expectations that homological properties of modules of finite projective dimension extend–in appropriate form–to all modules.

It is remarkable that several such conjectures have been refuted by using modules M, whose infinite minimal free resolution display the simplest numerical pattern: the graded Betti numbers $\beta^R_{i,j}(M)$ are zero for all $j \neq i$ (that is to say, M is *Koszul*), and $\beta^R_{i,i}(M) = p$ for some $p \geq 1$ and all $i \geq 0$; see [**11, 15, 16**]. Furthermore, in those examples both R and M have special properties: R is a *Koszul algebra*, meaning that k is a Koszul module, the Hilbert series $H_R(s) = \sum_{j\in\mathbf{Z}} \operatorname{rank}_k R_j s^j$ has the form $1 + es + (e-1)s^2$, and one has $H_M(s) = p + (e-1)ps$.

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