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A REFINEMENT OF SHARPLY F-PURE AND STRONGLY F-REGULAR PAIRS

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ABSTRACT. We point out that the usual argument used to prove that R is strongly F-regular if and only if R_Q is strongly F-regular for every prime ideal $Q \in \operatorname{Spec} R$, does not generalize to the case of pairs $(R, \mathfrak{a}^t)$. The author's definition of sharp F-purity for pairs $(R, \mathfrak{a}^t)$ suffers from the same defect. We therefore propose different definitions of sharply F-pure and strongly F-regular pairs. Our new definitions agree with the old definitions in several common contexts, including the case that R is a local ring.

1. Introduction. The notion of a strongly F-regular ring was introduced by Hochster and Huneke in [**10**] because it was easily seen to be well behaved with respect to localization (this is in contrast to weak F-regularity). It later was discovered that strongly F-regular rings (in characteristic $p > 0$) were closely related to rings with Kawamata log terminal singularities (in characteristic 0), see [**4, 7, 17**]. However, the notion of Kawamata log terminal singularities extends to pairs $(R, \mathfrak{a}^t)$ where $\mathfrak{a} \subseteq R$ is an ideal and $t > 0$ is a real number. Therefore, it was natural to ask whether there is an analogous notion of strong F-regularity for pairs $(R, \mathfrak{a}^t)$.

In [**16**], Takagi gave such a definition and proved that it satisfied many properties similar to Kawamata log terminal singularities (also see [**7, 19**]). In fact, by using this characteristic $p > 0$ definition, Takagi was able to prove remarkable results in characteristic zero for which there are still no known characteristic zero proofs, see for example [**16**, Theorem 4.1]. We now state this definition:

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