JOURNAL OF COMMUTATIVE ALGEBRA
Volume 1, Number 3, Fall 2009

PROPERTIES OF CUT IDEALS ASSOCIATED TO RING GRAPHS

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ABSTRACT. A cut ideal of a graph records the relations among the cuts of the graph. These toric ideals have been introduced by Sturmfels and Sullivant who also posed the problem of relating their properties to the combinatorial structure of the graph. We study the cut ideals of the family of ring graphs, which includes trees and cycles. We show that they have quadratic Gröbner bases and that their coordinate rings are Koszul, Hilbertian, and Cohen-Macaulay, but not Gorenstein in general.

1. Introduction. Let G be any finite (simple) graph with vertex set $V(G)$ and edge set $E(G)$. In [**15**] Sturmfels and Sullivant associate a projective variety X_G to G as follows. Let $A|B$ be an unordered partition of the vertex set of G. Each such partition defines a cut of the graph, denoted by $Cut(A|B)$, which is the set of edges $\{i,j\}$ such that $i \in A$, $j \in B$ or $j \in A$, $i \in B$. For each $A|B$, we can then assign variables to the edges according to whether they are in $Cut(A|B)$ or not. Thus, coordinates $q_{A|B}$ are indexed by unordered partitions $A|B$, while the variables encoding whether the edge $\{i,j\}$ is in the cut are s_{ij} and t_{ij} (for "separated" and "together"). The variety X_G, which we call *the cut variety of* G, is specified by the following homomorphism between polynomial rings:

$$\phi_G : K[q_{A|B} : A|B \text{ partition}] \to K[s_{ij}, t_{ij} : \{i,j\} \text{ edge of } G],$$
$$q_{A|B} \mapsto \prod_{\{i,j\}\in Cut(A|B)} s_{ij} \prod_{\{i,j\}\in E(G)\setminus Cut(A|B)} t_{ij}$$

The cut ideal I_G is the kernel of the map ϕ_G. It is a homogeneous toric ideal (note that $\deg \phi_G(q_{A|B}) = |E(G)|$). The variety X_G is defined by the cut ideal I_G.

The work for this paper was done while the first author was sponsored by the National Security Agency under Grant Number H98230-07-1-0065.
Received by the editors on June 3, 2008, and in revised form on August 19, 2008.
DOI:10.1216/JCA-2009-1-3-547