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EXTENDED MODULES

WOLFGANG HASSLER AND ROGER WIEGAND

Introduction. Suppose R and S are local rings and $(R, \mathbf{m}) \to (S, \mathbf{n})$ is a flat local homomorphism. Given a finitely generated S-module N, we say N is *extended* (from R) provided there is an R-module M such that $S \otimes_R M$ is isomorphic to N as an S-module. If such a module M exists, it is unique up to isomorphism (cf. [**9**, (2.5.8)]), and it is necessarily finitely generated.

The $\mathbf{m}$-adic completion $R \to \widehat{R}$ and the Henselization $R \to R^{\mathrm{h}}$ are particularly important examples. One reason is that the Krull-Remak-Schmidt uniqueness theorem holds for direct-sum decompositions of finitely generated modules over a Henselian local ring. Indeed, failure of uniqueness for general local rings stems directly from the fact that some modules over the Henselization (or completion) are not extended. Understanding which R^{h}-modules are extended is the key to unraveling the direct-sum behavior of R-modules.

Throughout, we assume that $(R, \mathbf{m})$ and $(S, \mathbf{n})$ are Noetherian local rings and that $R \to S$ is a flat local homomorphism. Many of our results generalize easily to a mildly non-commutative setting. Moreover, it is not always necessary to assume that our rings are local. Thus, we assume that A is a commutative ring, that B is a faithfully flat commutative A-algebra, and that Λ is a module-finite A-algebra.

Given a finitely generated left $B \otimes_A \Lambda$-module N, we say that N is *extended* (from Λ) provided there is a finitely generated left Λ-module M such that $B \otimes_A M$ is isomorphic to N as a $B \otimes_A \Lambda$-module.

In Sections 1 and 2 of the paper, we examine how the extended modules sit inside the family of all finitely generated modules. In Sections 3 – 4 we consider rings of dimension 2 and 1, respectively, find

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