JOURNAL OF COMMUTATIVE ALGEBRA
Volume 1, Number 1, Spring 2009

# COHEN-MACAULAY ADMISSIBLE CLUTTERS

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ABSTRACT. There is a one-to-one correspondence between square-free monomial ideals and clutters, which are also known as simple hypergraphs. In [**14**] it was conjectured that unmixed admissible clutters were Cohen-Macaulay. We prove that the conjecture is true for uniform clutters of heights 2 and 3, i.e., if the smallest cardinality of a minimal vertex cover of the clutter is 2 or 3. For clutters of greater height, we give a family of counterexamples to show that the conjecture fails. For unmixed admissible uniform clutters of height 4, we characterize when the Alexander dual of their edge ideals has linear quotients, and in particular, give an additional condition under which unmixed admissible uniform clutters are Cohen-Macaulay.

**1. Introduction.** A **clutter** consists of a finite set of points, called the **vertices**, and a family of nonempty subsets of the vertices with no nontrivial containments, called the **edges**. Clutters are also known as **simple hypergraphs**. A basic example of a clutter is a simple graph in the classical sense. Throughout the paper, $\mathcal{C}$ will denote a clutter over the vertices $V(\mathcal{C}) = \{x_1, \ldots, x_n\}$ and edges $E(\mathcal{C})$.

Let $K$ be a field. By identifying the vertices $\{x_1, \ldots, x_n\}$ with the variables of a polynomial ring $R = K[x_1, \ldots, x_n]$, there is a natural one-to-one correspondence between the class of clutters over the vertices $\{x_1, \ldots, x_n\}$ and the class of square-free monomial ideals in $R$. This correspondence is given by $\mathcal{C} \leftrightarrow I(\mathcal{C})$, where $I(\mathcal{C})$ is the ideal $\left\langle x^e = \prod_{x_i \in e} x_i \;\middle|\; e \in E(\mathcal{C}) \right\rangle$ in $R$. The ideal $I(\mathcal{C})$ is usually referred to as the **edge ideal** of $\mathcal{C}$. Edge ideals of clutters can also be viewed as edge ideals of hypergraphs (cf. [**8**]) or facet ideals of simplicial complexes (cf. [**4**]).

2000 AMS *Mathematics subject classification.* Primary 13F55, 05C65, 05C75.
*Keywords and phrases.* monomial ideals, edge ideals, Cohen-Macaulay, clutters, hypergraphs, Alexander dual, linear quotients
Received by the editors on January 31, 2008., and in revised form on March 5, 2008.
DOI:10.1216/JCA-2009-1-1-463 