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GRASSMANNIANS AND REPRESENTATIONS

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ABSTRACT. In this note we use Bott-Borel-Weil theory to compute cohomology of interesting vector bundles on sequences of Grassmannians.

1. Introduction. For any positive integer n the higher cohomology of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^{m-1}$ vanishes. The dimension of the space of global sections of this bundle is easily calculated to be $\binom{n+m}{n}$ via the identification of $H^0(\mathbb{P}^{m-1}, \mathcal{O}(n))$ with the vector space of homogeneous forms of degree n in m variables.

If we view $\mathbb{P}^{m-1}$ as the space of lines in an m-dimensional vector space V, then the line bundle $\mathcal{O}(n)$ is the n-th tensor power of the dual of the tautological line subbundle $\mathcal{O}(-1)$. Generalizing to the Grassmannian of k-planes we are led to a number of questions about the cohomology of vector bundles on Grassmannians.

The most obvious question is the following: Let V be a vector space of dimension m. Compute the dimension of the space of sections $H^0(\mathrm{Gr}(k, V), \mathcal{V}^{\otimes n})$ as a function of k, m, n, where $\mathcal{V}$ is the dual of the tautological rank k subbundle on the Grassmannian of k-planes in V. Likewise, if we are interested in the dimension of linear systems we could ask for the dimension of the linear system $H^0(\mathrm{Gr}(k, V), (\det \mathcal{V})^{\otimes n})$.

In this note we use Bott-Borel-Weil theory to answer these questions. In particular we will compute for any (irreducible) representation W of GL_k the dimension $H^0(\mathrm{Gr}(k, m), \mathcal{W})$ where $\mathcal{W}$ is the vector bundle on $\mathrm{Gr}(k, m)$ whose fiber at a point corresponding to a k-dimensional linear subspace L is the dual vector space W^* viewed as a $\mathrm{GL}(L)$-module. In addition we explain when the higher cohomology vanishes.

The results we obtain are variants of results that are well known in representation theory. See, for example, Section 4.1 of Weyman's book

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