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A GRÖBNER BASIS FOR THE SECANT IDEAL OF THE SECOND HYPERSIMPLEX

SETH SULLIVANT

ABSTRACT. We determine a Gröbner basis for the secant ideal of the toric ideal associated to the second hypersimplex $\Delta(2,n)$, with respect to any circular term order. The Gröbner basis of the secant ideal requires polynomials of odd degree up to n. This shows that the circular term order is 2-delightful, resolving a conjecture of Drton, Sturmfels, and the author. The proof uses Gröbner degenerations for secant ideals, combinatorial characterizations of the secant ideals of monomial ideals, and the relations between secant ideals and prolongations.

1. Introduction. If $X \subset \mathbb{P}^{m-1}$ is a projective variety, its rth secant variety $X^{\{r\}} \subseteq \mathbb{P}^{m-1}$ is the closure of the union of all planes in $\mathbb{P}^{m-1}$ spanned by r points in X. There is a large literature on secant varieties, and the vast majority of results focus on computing their dimension [**1, 2**]. Inspired by problems in computational complexity and algebraic statistics more attention has been paid to the problem of determining the vanishing ideals $I(X^{\{r\}})$ of secant varieties [**6, 8, 9**].

This paper presents a case study of the secant ideals $I(X^{\{2\}})$ of a particular family of toric varieties associated to the second hypersimplices

$$\Delta(2,n) = \mathrm{conv}(\{e_i + e_j \mid 1 \le i < j \le n\}).$$

The associated toric variety $X_{2,n}$ arises in algebraic geometry as the closure of the torus orbit of a generic point on the Grassmannian $Gr_{2,n}$. The secant varieties $X_{2,n}^{\{r\}}$ arise in statistics as the projectivization of the Zariski closure of the parameter space of the factor analysis model, with r-factors [**4**].

Our main result is the computation of a Gröbner basis for the secant ideal $I(X_{2,n}^{\{2\}})$, with respect to a certain circular term order, confirming

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