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THE DILWORTH LATTICE OF ARTINIAN RINGS

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ABSTRACT. After Freese [**3**] we define the Dilworth lattice of an Artinian local ring to be the family of ideals with the largest number of generators. We prove that it is indeed a lattice. Moreover we prove that over certain Gorenstein algebras the maximum and the minimum of the family are powers of the maximal ideal.

1. Introduction. In his paper [**8**] the second author defined the Dilworth number of Artinian rings and obtained some elementary results. It was followed by Ikeda [**4**] and [**5**]. These were analogs of some results in "combinatorial order theory" of finite sets as are found in e.g. [**1**, Chapter VIII].

The present paper was inspired by [**3**], where Freese showed that the family of antichains in a finite poset forms a lattice, hence it has the maximum and the minimum. The considerations made in the papers [**4**] and [**8**] suggest that an analogous result should be true for Artinian rings. An antichain in a poset may be interpreted in terms of commutative rings, as a minimal generating set of an ideal. Thus one may conceive that the family of ideals with the largest number of generators forms a lattice. This indeed is true and we prove it in Theorem 3. This was the starting point of this paper.

Easy examples show that these are basically infinite families even if we restrict the ideals to homogeneous ones. However, this leads us to considering the maximum and the minimum members of the lattice. It seems to be a natural question to ask under what conditions they are powers of the maximal ideal. A general result obtained in this paper on this question is Lemma 9. Under a certain condition of a Gorenstein algebra, we deduce that the Dilworth lattice has powers of the maximal ideal as the maximum and minimum. The condition

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