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GENERATING IDEALS IN PULLBACKS

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ABSTRACT. Let T be a domain, M a maximal ideal of T, $\varphi : T \to k = T/M$ the canonical projection, D a subring of the field k, and $R = \varphi^{-1}(D)$. We prove that if $I \nsubseteq M$ is an ideal of R for which $\varphi(I)$ can be generated by n elements of D and IT can be generated by m elements of T, then I can be generated by $\max\{2, n, m\}$ elements of R.

Consider the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \varphi\ } & k \end{array}$$

Here, T is a domain, M is a maximal ideal of T, $k = T/M$, $\varphi : T \to k$ is the canonical projection, D is a subring of k, and $R = \varphi^{-1}(D)$. Pullbacks of this type have frequently been used to provide important (counter-)examples for many years now. In [**3**] Gabelli and the present author discussed much of what is known about ideal theory in pullbacks. Although that work was primarily a survey, we did consider the problem of determining the number of generators of an ideal I of R from knowledge of the number of generators of $\varphi(I)$ in D and of IT in T. The purpose of this note is to give a complete solution to that problem. Our main result is

Theorem. *Let $I \nsubseteq M$ be an ideal of R such that $\varphi(I)$ is an n-generated ideal of D and IT is an m-generated ideal of T. Then I can be generated by* $\max\{2, n, m\}$ *elements of R.*

Since an r-generated ideal I of R both maps to an r-generated ideal of D and extends to an r-generated ideal of T, it is easy to see that this is

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