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PROPERTIES OF FACTORIZATIONS WITH SUCCESSIVE LENGTHS IN ONE-DIMENSIONAL LOCAL DOMAINS

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ABSTRACT. Let D be an atomic domain. Then every non-unit $a \in D\backslash\{0\}$ decomposes (in general in a highly non-unique way) into a product

$$a = u_1 \cdot \ldots \cdot u_n \tag{1}$$

of irreducible elements (atoms) u_i of D. The integer n is called the length of (1) and $\mathsf{L}(a) = \{n \in \mathbb{N} \mid \ a \text{ decomposes into } n \text{ irreducible elements of } D\}$ is called the set of lengths of a. Two integers $k < l$ are called successive lengths of a if $\mathsf{L}(a) \cap \{m \in \mathbb{N} \mid k \leq m \leq l\} = \{k, l\}$.

Suppose that D is a one-dimensional local domain with finite normalization. Then it is well known that $\Delta(D) = \{l - k \mid 0 \neq a \in D\backslash D^\times, k < l \text{ are successive lengths of } a\}$ is finite. Let $0 \neq a \in D$ be a non-unit and denote by $\mathsf{Z}_n(a)$ the set of factorizations of a with length n. In the present paper we investigate the structure of $\mathsf{Z}_n(a)$ and the relations between $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ if k and l are successive lengths of a. We prove that $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ are "similar" in a very strong sense except if k and l are close to the "boundaries" of $\mathsf{L}(a)$. We show by examples that in the latter case $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ may have a completely different structure. Finally, we apply our results to local quadratic orders of algebraic number fields.

1. Introduction. Let D be an integral domain. A nonzero non-unit $u \in D$ is called irreducible (or an atom) if u does not decompose into a product of two non-units of D. D is called atomic if every nonzero non-unit $a \in D$ has a factorization

$$a = u_1 \cdot \ldots \cdot u_n \tag{2}$$

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