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ON CHARACTERIZATIONS OF INTEGRALITY INVOLVING THE LYING-OVER AND INCOMPARABLE PROPERTIES

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ABSTRACT. The fact that residually algebraic pairs are the same as INC-pairs is generalized from the context of integral domains to that of arbitrary (commutative) rings. It is also shown that if $A \subseteq B$ are rings with D the integral closure of A in B, then B is integral over A if and only if (A, B) is an INC-pair for which the extension $D \subseteq B$ satisfies LO. However, a Noetherian local one-dimensional domain A is Henselian if and only if B is integral over A whenever B is a domain containing A such that (A, B) is an INC-pair for which the extension $A \subseteq B$ satisfies LO.

1. Introduction. All rings considered in this note are commutative with identity, and all subrings are unital. Following [**10**, page 28] we let LO, INC and GU denote the lying-over, incomparable and going-up properties for ring extensions. If $\mathcal{P}$ is a property of (some) ring extensions and $A \subseteq B$ are rings, we say that (A, B) is a *$\mathcal{P}$-pair* in case $D \subseteq E$ satisfies $\mathcal{P}$ for all rings $A \subseteq D \subseteq E \subseteq B$. The case of LO-pairs was introduced in [**5**], studied sporadically in the literature (cf. [**12**]), and recently given a new characterization in [**3**, Theorem 2.2]. It was shown in [**5**, Corollary 3.2] that GU-pairs are the same as LO-pairs. As for INC-pairs, they were introduced and characterized (without the terminology) in [**4**, Corollary 4] and studied further, but only in the context of extensions of (commutative integral) domains, in [**1**]. In particular, [**1**, Theorem 2.3] established that for extensions of domains, INC-pairs are the same as residually algebraic pairs. This domain-theoretic formulation has persisted in the summary of [**1**] given in the monograph [**8**], and several subsequent papers have also continued to study INC-pairs and residually algebraic pairs only for extensions of domains. Accordingly, our first order of business here is to generalize [**1**, Theorem 2.3] by showing that, for arbitrary ring extensions, the concepts of INC-pairs and residually algebraic pairs are equivalent.

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