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IDEALIZATION OF A MODULE

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ABSTRACT. Let R be a commutative ring and M an R-module. Nagata introduced the idealization $R\,(+)\,M$ of M. Here $R\,(+)\,M = R \oplus M$ (direct sum) is a commutative ring with product $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. The name comes from the fact that if N is a submodule of M, then $0 \oplus N$ is an ideal of $R\,(+)\,M$. The idealization can be used to extend results about ideals to modules and to provide interesting examples of commutative rings with zero divisors. We survey known results concerning $R\,(+)\,M$ and give some new ones too. The theme throughout is how properties of $R\,(+)\,M$ are related to those of R and M.

1. Introduction. Let R be a commutative ring with 1, and let M be a unitary R-module. Then $R\,(+)\,M = R \oplus M$ (direct sum) with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with 1 (even an R-algebra) called the *idealization of* M or the *trivial extension of* R *by* M. Note that R naturally embeds into $R\,(+)\,M$ via $r \to (r, 0)$, if N is a submodule of M, then $0\,(+)\,N$ is an ideal of $R\,(+)\,M$, $0\,(+)\,M$ is a nilpotent ideal of $R\,(+)\,M$ of index 2, and that $(R\,(+)\,M)/(0\,(+)\,M) \approx R$. Idealization is useful for (1) reducing results concerning submodules to the ideal case, (2) generalizing results from rings to modules and (3) constructing examples of commutative rings with zero divisors. The purpose of this article is to survey known results on idealization and to give some new ones and to give a history of the subject and its usefulness.

While we do not know who first constructed an example using idealization, the idea to use idealization to extend results concerning ideals to modules is due to Nagata. The preface to his famous book *Local rings* [**49**] states: "Among the new methods and new results given

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