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Invariant β and uniruled threefolds

Dedicated to Yoshikazu Nakai on his sixtieth birthday

By

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§ 0. Introduction.

In this paper, introducing a bimeromorphic invariant β, we shall make a systematic study of compact Kähler threefolds with $\kappa=-\infty$: For a compact complex manifold X, we put $\beta(X)=\mathrm{Max}\{\dim Y; Y$ is a compact complex manifold with $\kappa(Y)\geqq 0$ and there exists a generically surjective meromorphic map $f: X\to Y\}$. Then, if $\dim X\leqq 4$, one can naturally find a generically surjective meromorphic map $b_X: X\to B(X)$ such that $\beta(X)=\dim B(X)$ and that $f: X\to Y$ above always factors through $B(X)$, (cf. 1.1.3 and 1.1.4 of § 1). We now assume that X is a compact Kähler threefold. Then there exists a Zariski open dense subset U (resp. U') or $B(X)$ (resp. X) such that:

(a) $b_{X|U'}: U'\to b_X(U')=U$ is a proper morphism, and furthermore
(b) for every $u\in U$, the fibre $b_X^{-1}(u)$ is irreducible and $\beta(b_X^{-1}(u))=0$.

Thus, if $\beta(X)=1$ or 2, general fibres of b_X are rational. Therefore, it is naturally expected that problems of β can be translated into those of degenerations of rational curves or surfaces. In fact, via such translations, we shall prove:

(1) If $\pi: \tilde{X}\to X$ is a finite étale cover, then $\beta(\tilde{X})=\beta(X)$, (cf. Theorem 3.1.1).
(2) Let $g: Z\to S$ be a proper smooth morphism of Kähler manifolds such that $g^{-1}(s)=X$ for some $s\in S$. Assume that $\beta(X)=1$ or 2. Then $\beta(g^{-1}(s'))=\beta(X)$ for every $s'\in S$, (cf. Theorem 4.1.1).

Further results we obtained are the following:

(I) Let X be a compact Kähler uniruled threefold. Then
(I-a) $\beta(X)=0$ if and only if $q(X)=h^0(X, S^2(\Omega_X^2))=0$, (cf. Theorem 2.1.5);
(I-b) $\beta(X)=1$ if and only if $q(X)>h^0(X, S^{12}(\Omega_X^2))=0$;
(I-c) $\beta(X)=2$ if and only if $h^0(X, S^{12}(\Omega_X^2))\neq 0$;
(I-d) if $\pi: \tilde{X}\to X$ is a finite étale cover, then π naturally induces an étale cover $b(\pi): B(\tilde{X})\to B(X)$ with $\deg b(\pi)=\deg \pi$, (cf. Proposition 3.1.4);
(I-e) if $g: Z\to S$ is a proper smooth morphism of Kähler manifolds such that

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