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Finite multiplicity theorems for induced representations of semisimple Lie groups I

By

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Introduction

Let G be a connected semisimple Lie group with finite center, and $G=KA_pN_m$ be its Iwasawa decomposition. In his early work [5, I], Harish-Chandra proved that any irreducible quasi-simple (hence any irreducible unitary) representation π of G is admissible, that is to say, the restriction $\pi|K$ of π to the maximal compact subgroup K is of multiplicity finite. In view of the Frobenius reciprocity law, this theorem means that unitarily ($=L^2$-) or differentiably ($=C^\infty$-) induced representation $\mathrm{Ind}_K^G(\tau)$ has finite multiplicity property for any $\tau\in\hat{K}$, the unitary dual of K. Moreover, he obtained in [5, III, Theorem 4] an estimate of multiplicities in $\pi|K$ crucial for construction of the distribution character of π: there exists a constant $c_\pi>0$ such that, for any $\tau\in\hat{K}$,

$$\text{(0.1)} \qquad \dim \mathrm{Hom}_K(\pi, \tau)=\dim \mathrm{Hom}_G(\pi_\infty, C^\infty\text{-}\mathrm{Ind}_K^G(\tau))\leqq c_\pi \dim \tau\,,$$

where π_∞ is the smooth representation of G associated with π. (Actually $c_\pi=1$, see [5, II].) These theorems are obtained mainly through a careful study of infinite-dimensional representations of the Lie algebra $\mathfrak{g}$ of G from a purely algebraic point of view. Nevertheless, once the differentiability of K-finite vectors for π is established (the finite multiplicity theorem above assures the analyticity of such vectors), one can derive the important estimate (0.1) also by using the theory of (K, K)-spherical functions in [6], which is a purely analytical method.

In 1984, some parts of the latter analytical method were extended by E. P. van den Ban to the (K, H)-spherical functions for any semisimple symmetric pair (G, H). He proved in [1] that the induced representation $\mathrm{Ind}_H^G(1_H)$ has finite multiplicity property, where 1_H denotes the trivial one-dimensional representation of H. In another direction, M. Hashizume [7] studied (K, N_m)-spherical functions of special kind, so-called class one Whittaker functions. One of his results [7, Theorem 3.3], the finite-dimensionality of spaces of such functions, suggests us that the induced representation $\mathrm{Ind}_{N_m}^G(\xi)$ is of multiplicity finite for any one-dimensional representation (=character) ξ of the maximal unipotent subgroup N_m. (This is proved rigorously in § 4 of this paper.)

In the present article, we generalize the result of van den Ban, developing

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