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On first variation of Green's functions under quasiconformal deformation

By

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Introduction

The purpose of this note is to give variational formulas for Green's functions on arbitrary Riemann surfaces under quasiconformal deformation, which contains known formulas such as those due to Sontag [7], Guerrero [3] and Maitani [5].

After some preliminary discussion on quasiconformal mappings in § 1, we will prove the main formulas in § 2 and § 3 (Theorems 2 and 3).

§ 1. A surgery of quasiconformal mappings

Let U be the unit disk $\{|z|<1\}$ and $U'=\{|z|<r<1\}$. Then we can define a surgery of a given quasiconformal mapping f of U onto itself such that $f(0)=0$ as follows.

Let $\mu=\mu_f$ be the complex dilatation of f, and first decompose f as $f_2\circ f_1$ with quasiconformal mappings $f_1=f_1^{\mu}$ and $f_2=f_2^{\mu}$ of U onto itself such that the complex dilatations μ_{f_1} and μ_{f_2} are equal to $\mu|_{(U-U')}$ and $(\mu|_{U'}\cdot(f_1)_z/(\bar{f}_1)_{\bar{z}})\circ(f_1)^{-1}$, respectively, $f_1(0)=0$ (hence $f_2(0)=0$) and $f_2(1)=1$.

Next let H be the upper half plane in C, and set

$$\pi(z)=\exp(2\pi i\cdot z).$$

Then f_2 can be lifted to a quasiconformal mapping $F=F_2^{\mu}$ of H onto itself such that $F(0)=0$ and $\pi\circ F=f_2\circ\pi$. Since we can find a constant $r'<1$ depending only on r and a given $k\,(<1)$ such that $f_1(U')$ is contained in $\{|z|<r'\}$ whenever $\|\mu\|_\infty$ $(=\text{ess.}\sup_U|\mu|)$ $\leqq k$, F is conformal on $\{z\in H: 0<y<c\}$ with $c=(-1/2\pi)\cdot\log r'$, where $z=x+iy$. Here we may also assume that $c<1$.

Now set

$$\begin{aligned} F^\circ(z)&=F(z) \qquad \text{on } \{0<y<c/3\},\\ &=\frac{y-(c/3)}{c/3}\cdot z+\frac{(2c/3)-y}{c/3}\cdot F(z) \qquad \text{on } \{c/3\leqq y\leqq 2c/3\},\\ &=z \qquad \text{on } \{2c/3<y\}.\end{aligned}$$

Then clearly $F^\circ(0)=0$ and $F^\circ(z+1)=F^\circ(z)+1$, hence F° can be projected to a self-mapping $S(f_2)$ of $\bar{U}$ fixing 0 and 1. And setting $S(f)=S(f_2)\circ f_1$, we have a reformation of f. Note that $S(f)$ is conformal on U'. Moreover we can show the following

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