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Local existence for the semilinear Schrödinger equations in one space dimension

By

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1. Introduction

In this paper we study the initial value problem for the semilinear Schrödinger equations in one space dimension:

$$u_t - iu_{xx} = F(u, u_x) \quad \text{in} \quad (0, \infty) \times \mathbf{R}, \tag{1.1}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbf{R}, \tag{1.2}$$

where $u(t, x)$ is complex-valued, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, and $u_{xx} = \partial^2 u/\partial x^2$. We assume that the nonlinear term $F: \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is

$$F(u, q) \in C^\infty(\mathbf{R}^2 \times \mathbf{R}^2; \mathbf{C}), \quad |F(u, q)| \leq C(|u|^2 + |q|^2) \quad \text{near} \quad (u, q) = 0. \tag{1.3}$$

We regard the second variable q as u_x. Let $\partial/\partial u = 1/2(\partial/\partial v - i\partial/\partial w)$, $\partial/\partial \bar{u} = 1/2(\partial/\partial v + i\partial/\partial w)$, $\partial/\partial q = 1/2(\partial/\partial \xi - i\partial/\partial \eta)$, and $\partial/\partial \bar{q} = 1/2(\partial/\partial \xi + i\partial/\partial \eta)$ where $u = v + iw$, $q = \xi + i\eta$ and $v, w, \xi, \eta \in \mathbf{R}$.

The purpose of this paper is to show the local existence of solutions to (1.1) – (1.2). When we try to get a classical energy estimate, Im $\partial F/\partial q(u, u_x)$ which is imaginary part of coefficient of u_x gives the loss of derivatives, and then we cannot derive the estimate. Our idea to resolve this difficulity comes from the theory of linear Schrödinger type equations. More precisely, let us consider the following linear problem:

$$u_t - iu_{xx} + b(x)u_x + c(x)u = f(t, x) \quad \text{in} \quad \mathbf{R} \times \mathbf{R}, \tag{1.4}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbf{R}, \tag{1.5}$$

where $b(x), c(x) \in \mathscr{B}^\infty(\mathbf{R})$, $u_0(x) \in L^2(\mathbf{R})$, and $f(t, x) \in L^1_{\text{loc}}(\mathbf{R}; L^2(\mathbf{R}))$. According to Takeuchi [10] (see also Mizohata [6]), a necessary and sufficient condition for L^2-wellposedness to (1.4) – (1.5) is

$$\sup_{x \in \mathbf{R}} \left| \int_0^x \operatorname{Im} b(y)\, dy \right| < +\infty. \tag{1.6}$$

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