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On maps from BS^1 to classifying spaces of certain gauge groups II

By

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1. Introduction

The purpose of this paper is to generalize Theorem 1.2 of [5]. Let π: $P \to X$ be a principal SU(2) bundle over a simply connected closed 4 manifold X and $\mathcal{G}$ its gauge group. $\mathcal{G}$ is identified with $\Gamma(AdP)$, all continuous sections of the adjoint bundle of P, and we give the compact open topology on it.
We show the following result.

Theorem 1.1 *The following three conditions are equivalent.*
1. *There exists a homotopically non trivial map from BS^1 to $B\mathcal{G}$.*
2. *There exists a non trivial homomorphism from S^1 to $\mathcal{G}$.*
3. *The structure group of P reduces to S^1.*

Remark 1.2 In [5], we showed this result under the assumption that X is a smooth simply connected spin 4 manifold or $\mathbf{C}P^2$.

It is clear that *3* implies *2* and by the Appendix of [5], *2* implies *1*. The structure group of P reduces to S^1 if and only if there exists an element $u \in H^2(X)$ such that $c_2(P) = -u^2[X]$. We will show that *1* implies $c_2(P) = -u^2[X]$. In this paper $H_*(\ \ )$ $((H^*))$ mean the integral (co) homology.

2. Proof of Theorem 1.1

Note that principal SU(2) bundles over X are classified by their 2nd Chern classes. If $c_2(P) = k$, by [1], we have a homotopy equivalence

$$B\mathcal{G} \simeq \mathrm{Map}_k(X, B\mathrm{SU}(2)),$$

where $\mathrm{Map}_k(X, B\mathrm{SU}(2))$ denotes the connected component of $\mathrm{Map}(X, B\mathrm{SU}(2))$ containing the map inducing P and a fibration

$$\mathrm{Map}_k^*(X, B\mathrm{SU}(2)) \to \mathrm{Map}_k(X, B\mathrm{SU}(2)) \xrightarrow{ev} B\mathrm{SU}(2),$$

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