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On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms

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Introduction

The purpose of this paper is twofold. The first one is to discuss various group topologies on inductive limits of topological groups, and unitary representations of inductive limit groups in a certain case, and the second one is to treat group topologies in the case of the group of diffeomorphisms.

Contrary to the affirmative statement in [1] or in [5], the inductive limit of topologies of an inductive system of topological groups does not always give a group topology, or more exactly, the multiplication is not necessarily continuous with respect to the inductive limit topology (denoted by τ_{ind}). In Part I of this paper, we show this by a simple example in the case of abelian groups, and then discuss in general which kinds of group topologies can be chosen on an inductive limit group under the condition that they are weaker than τ_{ind}.

We study in particular the case where inductive system is countable and essentially consists of locally compact groups. (For exact definition, see §2.3, and such a system is called a *countable LCG inductive system* in short). Then we prove that the inductive limit topology τ_{ind} gives a group topology in this case (Theorem 2.7), and also that it is essentially a unique one under a mild condition (Theorem 5.6).

Further, for a countable LCG inductive system, we discuss in a certain extent unitary representations and continuous positive definite functions of the inductive limit group $G = \varinjlim G_j$, and prove that, under the same condition as for Theorem 5.6, there exist sufficiently many of them so that the points of G can be separated (Theorem 5.7). Since there does not exist in general a Haar measure on G, the important point of the discussion is the limiting process from the case of locally compact groups G_j to G.

In Part II, we discuss the case of the group $G = \mathrm{Diff}_0(M)$ of diffeomorphisms with compact supports on a connected, non-compact, C^r-manifold M, $1 \leq r \leq \infty$, and prove that the inductive limit topology τ_{ind} never gives a group topology (Theorem 6.1).

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