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On a property of Nirenberg type operator

By

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§ 1. Introduction

Let X be a nowhere-zero C^∞ complex vector field in R^n. Let $S^X = \{f \in C^\infty(R^n);\ Xu = f$ has a C^1 solution near the origin.$\}$ and $S_X = \{f \in C^\infty(R^n);\ Xu = f$ has a C^1 solution near the origin such that $du(0) \neq 0\}$.

The following facts are classically well known:

(1) $\mathscr{A} \subset S_X$ if X is real-analytic, where $\mathscr{A}$ denotes the set of real-analytic functions in R^n.

(2) $S_X = C^\infty(R^2)$ if $n = 2$, and $X(0)$, $\bar{X}(0)$ are C-linearly independent (In this case, X is an elliptic operator).

And we can easily obtain the following fact owing to Hölmander [1] and Treves [4]:

(3) $S_X = C^\infty(R^n)$ if X is a solvable operator at the origin.

Though it is trivial, we also know the following fact:

(4) $\mathscr{A} \subset S_X \subsetneq C^\infty(R^n)$ if X is a non-solvable operator at the origin and real-analytic.

We thus see $S_X = S^X$ in each case of the above. Does there exist a non-solvable vector field X such that $S_X \subsetneq S^X$?

This paper aims at showing that the answer is "Yes". We shall give such vector fields L_α, which we call Nirenberg type:

Let $\alpha(t,x)$ be a real-valued $C^\infty(R^2)$ function satisfying the following conditions:

(A.1) *$\alpha(t,x) \geq 0$ in a neighborhood ω of the origin.*

(A.2) *There exist positive constants c, d, and a monotonously increasing sequence $\{p_n\}$ of positive integers such that*

$$\iint_{D(p_k)} \alpha(t,x)\,\mathrm{d}t\mathrm{d}x > \frac{9}{(p_k+d)(p_k+c)}$$

for every sufficiently large k, where $D(p_k) = \left(0, \frac{1}{p_k}\right) \times \left(0, \frac{1}{p_k}\right)$.

We shall define L_α in the following manner:

$$L_\alpha = \partial_t + i(2t+\alpha)\partial_x.$$

Then we assert the following

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