

Blow-ups of $\mathbf{P}^2$ and root systems of type D

By

Jun-ichi MATSUZAWA and Akiko OMURA

1. Introduction

Nonsingular cubic surfaces in $\mathbf{P}^3(\mathbf{C})$ are obtained by blowing up 6 points on $\mathbf{P}^2$. Also it is well known that geometry of cubic surfaces is closely related to the root system and Weyl group of type E_6 : (i) The symmetry of the 27 lines on nonsingular cubic surface can be described by the root system and Weyl group of type E_6 . (ii) In the middle homology lattice of cubic surface, the orthogonal complement of the class of canonical divisor is isomorphic to the root lattice of type E_6 . (iii) The semi-universal deformation of simple singularity of type E_6 can be described by a Cartan subalgebra of Lie algebra of type E_6 . Furthermore a nonsingular cubic surface can be regarded as a compactification of a generic fiber of this deformation.

For certain class of rational surfaces, the geometry of surfaces is closely related to infinite root systems and the moduli space for the surfaces are constructed in terms of root systems and periods [9].

In this paper, we construct rational surfaces related to the root system and Weyl group of type D_m . We discuss the moduli problem of the surfaces.

In sections 2 and 3, we show the relation between surfaces X_m obtained by blowing up m points on $\mathbf{P}^2$ and the root systems and Weyl groups of type D_m . In section 4, we prove the theorem of Torelli type for the pairs of X_m and a certain anticanonical divisor of X_m in terms of the root systems and Weyl groups of type D_m . In section 5, we construct a family $\varphi: \mathfrak{X} \rightarrow S$ of the surfaces X_{2n+3} , where the base space S is the quotient space of the Cartan subalgebra of simple Lie algebra of type D_{2n+3} by its Weyl group.

The nonsingular fiber $\mathfrak{X}_s$ can be regarded as a compactification of the fiber of semi-universal deformation of the simple singularity of type D_{2n+3} . So the relation between X_m and the simple singularity of type D_{2n+3} is similar to that between Del Pezzo surfaces and the simple singularities of type E (see Remark 5.10). In section 6, we show the relation between the surface Z_{2n+2} obtained by blowing up X_{2n+2} and the root system of type D_{2n+2} . Also we can construct a family $\varphi: \mathfrak{X} \rightarrow S$ of these surfaces Z_{2n+2} , where the base space S is the quotient space of the Cartan subalgebra of simple Lie algebra of type D_{2n+2} by its Weyl group. The