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On the bamboo-shoot topology of certain inductive limits of topological groups

By

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§ 0. Introduction

Let $\{(G_n\tau_n), \phi_{n+1\,n}\}_{n\in N}$ be an inductive system of topological groups G_n with topology τ_n, each $\phi_{n+1\,n}$ being a continuous homomorphism of G_n into G_{n+1}. Put $G = \varinjlim G_n$ and $\tau_{\text{ind}} = \varinjlim \tau_n$. N. Tatsuuma—H. Shimomura—T. Hirai [2] showed by two counter examples that τ_{ind} is not necessarily a group topology for G. They also showed that if the given inductive system fulfils the "PTA-condition", there exists for G the finest group topology that makes every canonical map ϕ_n of G_n into G continuous. Such a topology is, of course, coarser than τ_{ind}. They called such a topology the bamboo-shoot topology for G, denoted by τ_{BS}, and gave a τ_{BS}-neighbourhood base at the unity e of G as the collection of all sets

$$U[k] = \bigcup\nolimits_{n \geqq k} \phi_n(U_n)\phi_{n-1}(U_{n-1}) \cdots \phi_k(U_k)\phi(U_k) \cdots \phi_{n-1}(U_{n-1})\phi_n(U_n)$$

with $k = 1, 2, \ldots$ and U_j's each of which runs over symmetric neighbourhoods of the unity e_j of (G_j, τ_j), $j \geqq k$. Here the PTA-condition is a moderate one and stated as follows:

$$(0.1) \qquad \forall n,\ \forall U,\ \exists V \subseteq U, \quad V = V^{-1}, \quad \forall m > n,\ \forall W,\ \exists W', \\ W'\phi_{mn}(V) \subseteq \phi_{mn}(V)W,$$

where U, V (resp. W, W') denote neighbourhoods of the unity e_n of G_n (resp. e_m of G_m) and $\phi_{mn} = \phi_{mm-1} \circ \cdots \circ \phi_{n+1\,n}$. For instance, any inductive system consisting of locally compact Hausdorff groups fulfils this condition and in this case τ_{ind} happens to coincide with τ_{BS}. τ_{BS} in general seems to be a topological- group-theoretic analogue of the locally convex inductive topology of the inductive limit of locally convex vector spaces (see Propositions 3.1 and 3.2 in [2]).

Now let us bring an inductive system of Banach algebras A_n $(n \in N)$ with the limit algebra $A = \varinjlim A_n$ (in algebraic sense). Let τ_{lct} denote the locally convex inductive topology of A as the inductive limit of Banach spaces A_n. In an appropriate circumstance this system yields an inductive system of topological

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