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A necessary and sufficient condition of local integrability

By

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§1. Introduction

Let X_n be a nowhere-zero C^∞ complex vector field defined near a point P in $\mathbf{R}^n$. We shall say that X_n is locally integrable at P if the equation $X_n u = 0$ has C^1 solutions $u_1, u_2, \dots, u_{n-1}$ near P such that $du_1 \wedge du_2 \wedge \cdots \wedge du_{n-1}(\mathrm{P}) \neq 0$ (see [3], [4], [5], and [12]).

Generally, the following is known: X_n is locally integrable at P if X_n is real-analytic or locally solvable at P (see [14], for instance) but there exist non-solvable vector fields which have no local integrability (due to Nirenberg [9]).

In this article we are concerned with the case where $n = 2$.

The equation $X_2 u = 0$ near P can be transformed into that of the form

$$Lu \equiv (\partial_t + ia(t,x)\partial_x)u = 0$$

near the origin in $\mathbf{R}^2$, where $a(t,x)$ is a real-valued C^∞ function.

Though there are several partial results ([7], [8], [10], [11], [12], [13], [14] for instance), the problem is open to get a necessary and sufficient condition for $Lu = 0$ to have a solution near the origin such that $\partial_x u \neq 0$:

Suppose that $a(t,x)$ is real-analytic with respect to x. Then the equation $Lu = 0$ has such a solution by the existence theorem of Cauchy-Kovalevskaya-Nagumo.

So let $a(t,x)$ be not real-analytic with respect to x. In the case where the function $t \to a(t,x)$ does not change sign in $\{t; (t,x) \in \mathcal{O}\}$ for every x by taking a neighborhood $\mathcal{O}$ of the origin, we see that the equation $Lv = -ia_x(t,x)$ has a C^∞ solution v near the origin by the local solvability of L; thus we find that the function

$$\int_0^t -ia(\xi,x)\exp\{v(\xi,x)\}d\xi + \int_0^x \exp\{v(0,\eta)\}d\eta$$

is one of the solutions satisfying the equation $Lu = 0$ with $\partial_x u \neq 0$ near the origin.

In the last case where the function $t \to a(t,x)$ changes sign in $\{t; (t,x) \in \mathcal{O}\}$ for some x by taking any small neighborhood $\mathcal{O}$ of the origin, there exists an example

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