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Finsler geometry of projectivized vector bundles

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Introduction

The purpose of this article is to reformulate the algebraic geometric concept of ampleness (and the numerical effectiveness) of a holomorphic vector bundle E in terms of Finsler geometry (cf. also Aikou [2], [3]; for general theory on Finsler geometry we refer to [5], [6] and [1]). We also provide some implications of this reformulation. For applications of the formulation using projectivized bundles to complex analysis see [7], [8] and [17]. As expected, the condition involves the concept of a Finsler metric along the fibers of E. By a (complex) Finsler metric (see Section 4 for more details) on E we mean a non-negative function h on E with the following properties:

(FM1) *h is an upper semi-continuous function on E;*
(FM2) *$h(z, \lambda v) = |\lambda| h(z, v)$ for all $\lambda \in \mathbf{C}$ and $(z, v) \in E_z$;*
(FM3) *$h(z, v) > 0$ on $E \backslash \{$zero-section$\}$;*
(FM4) *for z and v fixed the function $h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.*

For example the Kobayashi metric on a hyperbolic manifold is a Finsler metric on the tangent bundle. More generally, any intrinsic (i.e., depending only on the complex structure) (pseudo)-metric of a complex manifold is a Finsler (pseudo)-metric (i.e., (FM3) is replaced by the weaker condition $h(z, v) \geq 0$ on E). Obviously the norm of a *Hermitian* metric on E is Finsler and satisfies, among others, the following additional conditions:

(FM5) *h is of class $\mathcal{C}^0$ on E and of class $\mathcal{C}^\infty$ on $E \backslash \{$zero-section$\}$;*
(FM6) *h is strictly pseudoconvex on $E_z \setminus \{0\}$ for all $z \in M$.*

This last two properties are, in general, not shared by the intrinsic metrics, e.g., the Kobayashi metric is not even continuous unless it is complete (so M is complete hyperbolic); and, even in the complete case, it is in general not smooth outside the zero section. On the other hand, as we shall see, there are many Finsler metrics with these additional properties which are not Hermitian. Without these last 2 conditions differential geometric concepts

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