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Characters of wreath products of finite groups with the infinite symmetric group

By

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Introduction

1. Let G be a countable discrete group, $K_1(G)$ the set of all positive definite class functions f on G normalized as $f(e) = 1$, and $E(G)$ the set of all extremal elements in the convex set $K_1(G)$, where e denotes the identity element of G. In [Tho1], a canonical bijective correspondence between $E(G)$ and the set of characters of all factor representations of finite type is established (cf. **1.2** below). In this sense every element $f \in E(G)$ is called a *character* of G.

The purpose of this paper is to give explicitly all the characters of the wreath product groups $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ of any finite groups T with the infinite symmetric group $\mathfrak{S}_\infty$. This problem of determining all the characters of factor representations of finite type, or the problem of giving a general character formula for $f \in E(G)$, was worked out in [Tho2] for $G = \mathfrak{S}_\infty$. The result for $G = GL(\infty, \boldsymbol{F}_q)$ with a finite field $\boldsymbol{F}_q$ was given in [Sk].

The case of infinite symmetric group attracted interests of many mathematicians and we cite here, among others, works of Vershik-Kerov [VK], Kerov-Olshanski [KO] and Biane [Bi] in which they worked principally from the point of view of approximation from finite symmetric groups $\mathfrak{S}_n$ $(n \to \infty)$. Recently in [Hi3]–[Hi4], we reexamined the case of $\mathfrak{S}_\infty$ from the standpoint of taking limits of centralizations of positive definite functions obtained as matrix elements of simple unitary representations. Since this is one of our main ideas, let us explain it briefly here. For a subgroup G' of G a *centralization* of a function F on G with respect to G' is by definition

$$F^{G'}(g) = \frac{1}{|G'|} \sum_{g' \in G'} F(g' g \, {g'}^{-1}) \qquad (g \in G).$$

Taking an appropriate series of increasing subgroups $G_N \nearrow G$ as $N \to \infty$, we consider pointwise limit $f(g) = \lim_{N\to\infty} F^{G_N}(g)$. Here as a function F, we choose a positive definite matrix element of an induced representation $\rho = \mathrm{Ind}_H^G \pi$ of a (not necessary irreducible) unitary representation π of a subgroup

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