COVERING OF MANIFOLDS WITH OPEN CELLS

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1. Introduction

We prove a fundamental theorem for manifolds.

THEOREM 1. *A connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with $n + 1$ open topological (piecewise linear, differentiable) cells.*

It is well known that a manifold can be covered with $n + 1$ coordinate systems [4].

Applying engulfing theorems, we can strengthen Theorem 1. If x is a real number, let $\langle x \rangle$ denote the least integer greater than or equal to x.

THEOREM 2. *Let $k \leq n - 3$. A k-connected n-dimensional topological (piecewise linear, differentiable) manifold without boundary can be covered with $\langle (n + 1)/(k + 1) \rangle$ open topological (piecewise linear, differentiable) cells.*

The statement of Theorem 2 for the cases $n = 3, k = 1$ and $n = 4, k = 2$ is for closed manifolds equivalent to the topological (piecewise linear, differentiable) version of the Poincaré conjecture in dimensions $n = 3$ and $n = 4$ respectively. The case $n = 2, k = 1$ is well known: A 1-connected 2-dimensional topological (piecewise linear, differentiable) manifold without boundary is homeomorphic (piecewise linearly homeomorphic, diffeomorphic) to the 2-sphere if it is compact, and to the 2-dimensional euclidean space if it is not compact [7].

Theorem 1 and 2 hold also for analytic manifolds and coverings with open analytic cells. This follows from Theorem B of [6].

The theorems are proved by a certain technique which was motivated by the proof of the topological Poincaré conjecture in dimensions $n \geq 5$ in [5]. Our arguments apply simultaneously to topological, piecewise linear, and differentiable manifolds. The differentiable version of the theorems can also be obtained directly from the piecewise linear results in the fashion of [1], introducing smooth triangulations and approximating piecewise linear homeomorphisms by diffeomorphisms.

Theorem 2 also improves a result in [10].

I would like to thank the referee for his comment and Mr. MacLean for a remark.

2. Preliminaries

By an n-dimensional topological manifold M^n without boundary, we mean a separable Hausdorff space such that each point of M^n has an open neighbor-

Received August 10, 1967.