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UNIQUENESS OF THE MAXIMAL FUNCTION IN THE RATIO ERGODIC THEOREM

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ABSTRACT. We show that the maximal operator associated to Hopf's ratio ergodic theorem is injective.

1. Introduction

In a recent paper L. Ephremidze has shown that for a measure preserving transformation (m.p.t.) T on a finite measure space $(X, \mathcal{A}, \mu)$ the *ergodic maximal function* $M(f) := \sup_{n\geq 1} n^{-1}\mathbf{S}_n(f)$, where $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$, uniquely determines $f \in L_1(\mu)$, i.e., $M(f) = M(g)$ a.e. implies $f = g$ a.e., cf. [E]. (An alternative short proof on this result has been given in [J].)

His article also discusses to what extent this remains true if the measure space is infinite (but σ-finite), proving that the conclusion still holds for non-negative functions, and showing that it does break down for some others. While this observation certainly is of some interest, one might argue that in infinite measure preserving situations (see [A]), $M(f)$ is not the "correct" object to study (there being no nontrivial limiting behaviour of $n^{-1}\mathbf{S}_n(f)$). Instead, we are going to consider the maximal function corresponding to the proper version of the pointwise ergodic theorem for infinite measure spaces, that is, to Hopf's ratio ergodic theorem (cf. [S], [H]). We briefly recall the statement of the latter (see [KK] and [Z] for short proofs):

THEOREM 1 (Hopf's Ratio Ergodic Theorem). *Let T be a conservative m.p.t. on the $\sigma-$finite measure space $(X, \mathcal{A}, \mu)$. Let $f, p \in L_1(\mu)$ with $p > 0$. Then there exists a measurable function $Q(f, p) : X \to \mathbb{R}$ such that*

$$\frac{\mathbf{S}_n(f)}{\mathbf{S}_n(p)} = \frac{\sum_{k=0}^{n-1} f \circ T^k}{\sum_{k=0}^{n-1} p \circ T^k} \longrightarrow Q(f,p) \quad \text{a.e. on } X \quad \text{as } n \to \infty.$$

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