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ON TORSION-FREE GROUPS IN O-MINIMAL STRUCTURES

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ABSTRACT. We consider groups definable in the structure $\mathbb{R}_{an}$ and certain o-minimal expansions of it. We prove: If $\mathbb{G} = \langle G, * \rangle$ is a definable abelian torsion-free group, then $\mathbb{G}$ is definably isomorphic to a direct sum of $\langle \mathbb{R}, + \rangle^k$ and $\langle \mathbb{R}^{>0}, \cdot \rangle^m$, for some $k, m \geqslant 0$. Futhermore, this isomorphism is definable in the structure $\langle \mathbb{R}, +, \cdot, \mathbb{G} \rangle$. In particular, if $\mathbb{G}$ is semialgebraic, then the isomorphism is semialgebraic.

We show how to use the above result to give an "o-minimal proof" to the classical Chevalley theorem for abelian algebraic groups over algebraically closed fields of characteristic zero.

We also prove: Let $\mathcal{M}$ be an arbitrary o-minimal expansion of a real closed field R and $\mathbb{G}$ a definable group of dimension n. The group $\mathbb{G}$ is torsion-free if and only if $\mathbb{G}$, as a definable group-manifold, is definably diffeomorphic to R^n.

1. Introduction

Throughout this paper we fix an o-minimal expansion $\mathcal{M}$ of a real closed field $R = \langle R, +, \cdot, 0, 1, < \rangle$. By "definable" we always mean definable in $\mathcal{M}$.

It is well-known that every abelian connected real Lie group is Lie isomorphic to a direct sum of copies of $\mathbb{R}_a$ and the circle group S^1 (see, for example, [2]). Here and everywhere below for a real closed field R we will denote by R_a its additive group $\langle R, +, 0 \rangle$, and by R_m the multiplicative group of positive elements $\langle R^{>0}, \cdot, 1 \rangle$.

From a model-theoretical point of view it is natural to ask whether or not this kind of decomposition holds in the category of groups definable in the o-minimal structure $\mathcal{M}$.

In general, the answer to the above question is negative. There are at least two obstacles. First, in the polynomial bounded case, the multiplicative group R_m is not definably isomorphic to the additive group of R_a, and one should at least allow also copies of the multiplicative group.

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