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SPECTRAL PROPERTIES OF TRANSLATION OPERATORS IN CERTAIN FUNCTION SPACES

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0. Introduction

Let G be a locally compact abelian group and $1 \leq p \leq \infty$, $p \neq 2$. Then, except in trivial cases, translation operators in $L^p(G)$ are not spectral operators in the sense of N. Dunford [3]; see [2; Chapter 20] and [6], for example. However, it is natural to expect translations to be spectral in some sense because they are isometries of $L^p(G)$ onto itself and, hence, analogues of unitary operators in Hilbert space. For $1 < p < \infty$, this point was taken up in [1; §4] and [6], where it is shown that translations can indeed be expressed in the form

$$\int_{\mathbf{R}} e^{i\lambda}\, dQ(\lambda), \tag{1}$$

where $\{Q(\lambda); \lambda \in \mathbf{R}\}$ is an associated spectral family of commuting projections satisfying certain properties, [1; §4], and the "integral" (1), which can be interpreted as being over the unit circle $\mathbf{T}$ of the complex plane $\mathbf{C}$, exists in a certain well defined sense. On the other hand it should be stressed that in general the spectral family does not generate a σ-additive, projection-valued spectral measure.

However, as shown in the recent note [5], once it is realized that translation operators fail to be spectral for two very different reasons, then in many cases an alternative interpretation of (1) is possible. It may happen that the operator fails to be spectral simply because its domain space is "too small" to accommodate the projections needed to form its resolution of the identity. Accordingly, if interpreted as acting in a suitable space containing the domain space, it often happens that the operator is spectral in Dunford's sense. This has the advantage that the operator then has associated with it a rich functional calculus. For example, this is always the case if G is any locally compact abelian group and $1 < p \leq 2$, or if G is compact and $1 < p < \infty$ or if $2 < p < \infty$ and the element $g \in G$ defining the translation operator generates a compact metrizable subgroup of G; see [5]. In contrast, it is also the case, for fundamentally different reasons, that there exist non-trivial translations in $L^p(G)$, $2 < p < \infty$, for certain groups G, which are *genuinely*

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