THE SPACE OF HOMEOMORPHISMS ON A TORUS[1]

BY

MARY-ELIZABETH HAMSTROM

There have been several recent results concerning homotopy properties of the space of homeomorphisms on a manifold. Most of these properties have been local. In [4], Eldon Dyer and I proved that the space of homeomorphisms on a 2-manifold is locally contractible and in [5] and [6] it is proved that the space of homeomorphisms on a 3-manifold is locally homotopy connected in all dimensions. Global properties appear to be more difficult. A well known result of Alexander's [1] states that the space of homeomorphisms on an n-cell leaving its boundary pointwise fixed is contractible and locally contractible. In a recent paper [7] it is proved that the *identity component* of the space of homeomorphisms on a disc with holes leaving its boundary pointwise fixed is homotopically trivial. In the present paper, the identity component of the space of homeomorphisms on a torus is considered and it is proved that its homotopy groups are the same as those for the torus. For related results, see [2], [11], [12], and [13].

THEOREM 1. *If k is an integer greater than* 1, *then the identity component of the space H of homeomorphisms of a torus T onto itself has the property that* $\pi_k(H) = 0$.

Proof. Let C denote a meridian simple closed curve on T and P a point of C. A covering space of T is $C \times E^1$, where E^1 is the real line and the covering map π is such that $\pi(x, 0) = x$ for each x in C and, in general, $\pi(x, t) = \pi(y, t')$ if and only if $x = y$ and $t - t'$ is an integer. If n is a non-negative integer, S^n denotes an n-sphere and will be considered as the boundary of the $(n + 1)$-cell, R^{n+1}.

Let F denote a mapping of S^k into H and g the mapping of S^k into T defined by $g(x) = F(x)(P)$. There exists a mapping G of S^k into $C \times E^1$ such that $\pi G(x) = g(x)$ and for each x in S^k, there is a unique mapping $f(x)$ of C into $C \times E^1$ such that $f(x)(P) = G(x)$ and for y in C, $\pi f(x)(y) = F(x)(y)$. The existence of G is a consequence of the various lifting properties of fiber spaces. (See [10, p. 63, Th 3.1.].) To see that $F(x) \mid C$ can be lifted, note that $F(x) \mid C$ is homotopic to the identity in T, since F is in the identity component of H. In particular, there is a mapping φ of $C \times I$ into T such that $\varphi \mid C \times 0$ is a homeomorphism onto a meridian of T, $\varphi \mid C \times 1 = F(x)$ and $\varphi(P, t) = g(x)$. (See Lemma A.) Since $C \times 0$ is a strong deformation retract of $C \times I$ and there is clearly a mapping $\tilde{\varphi}$ of $C \times 0$ into $C \times E^1$ such

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