ON A PROBLEM OF STÖRMER

BY

D. H. LEHMER

1. Introduction

Let $q_1 < q_2 < \cdots < q_t$ be a given set of t primes, and let Q be the set of all numbers

$$q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_t^{\alpha_t} \qquad (\alpha_i \geqq 0, \quad i = 1(1)t)$$

generated by these primes. We consider the question of finding pairs $(S, S + 1)$ of consecutive integers such that both S and $S + 1$ belong to Q. Since it is obvious that no such pair exists unless $q_1 = 2$, we are at the same time asking about those members of Q which are triangular numbers. Interest in such pairs dates back to the 18$^{\text{th}}$ century and seems to have been awakened by their usefulness in calculating logarithms of integers to great accuracy. Gauss notes for example that

$$9800 = 2^3 \cdot 5^2 \cdot 7^2, \qquad 9801 = 3^4 \cdot 11^2.$$

Such pairs lead to sets of "nearly" dependent logarithms of primes. For instance the number

$$\begin{aligned} K &= \log 63927525376 - \log 63927525375 \\ &= 13 \log 2 - 3 \log 3 - 3 \log 5 - 7 \log 7 \\ &\qquad\qquad + 4 \log 11 + \log 13 - \log 23 + \log 41, \end{aligned}$$

which cannot be zero because of the unique factorization theorem, is, however, less than $1.56427 \cdot 10^{-11}$.

Another use for such pairs is in finding particular solutions of diophantine equations of the form

$$Ax^n - By^m = 1.$$

For example the equation

$$x^2 - 14y^3 = 1$$

has the solution (55, 6) because of the pair (3024, 3025). In a recent proof of some results on the distribution of consecutive pairs of higher residues, many hundreds of such pairs were used with t ranging up to 32 [1].

The problem proposed and solved by Størmer [2] is that of finding *all* pairs $(S, S + 1)$ both belonging to the given set Q. He showed that there are indeed only a finite number of such pairs, and that they can be found in a nontentative way by solving $3^t - 2^t$ Pell equations. He gave all 23 pairs that go with the set

$$Q : 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4}.$$

Received July 25, 1962.