ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 1, Spring 1983

FINE CONVERGENCE AND PARABOLIC CONVERGENCE FOR THE HELMHOLTZ EQUATION AND THE HEAT EQUATION

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Introduction

Consider, for $\alpha > 0$ fixed, the Helmholtz equation $\Delta u - 2\alpha u = 0$ on $\mathbf{R}^n$. It is not hard to see that the corresponding Martin boundary is a sphere, and that every positive solution has an integral representation

$$u(x) = \int_{S^{n-1}} e^{\lambda\langle x, b\rangle}\mu(db)$$

where $\lambda = (2\alpha)^{1/2}$ and μ is a positive measure on the sphere. The rotationally invariant function given by this formula where μ is equal to Lebesque measure σ will be denoted by h.

O. Linden [12] proved a Fatou-type theorem recovering the values of $d\mu/d\sigma$ a.e. as limits at infinity of u/h along tubes of constant diameter. The present article proves a stronger result. It gives convergence through parabolic regions, which are in many ways more natural than tubes, and it does not require u to be globally defined. In other words this is an analogue of the well-known result of Privalov-Calderón-Carleson [2], [3] about harmonic functions in a half space of $\mathbf{R}^n$.

The method of proof is essentially that of Brelot and Doob [1], also used in [10]; it consists in deriving a geometric convergence result from fine convergence at the Martin boundary, which is guaranteed in a very general situation by the Fatou-Naïm-Doob theorem. There is however an essential difficulty to be surmounted: the natural version of the Harnack inequalities for the associated potential theory is not strong enough to permit the direct translation of the argument of Brelot and Doob to the Helmholtz equation (see remark following definition 2.1). In order to bypass this difficulty it is first useful to make a not entirely trivial reduction of the problem (Theorem 1.2) and then to use a strengthened one-sided Harnack type inequality (Proposition 2.4) which is obtained from the theory of the heat equation.

In Section 1 the reduction theorem is proved and section two gives the

Received February 9, 1981.
[1] Partially supported by the National Science Foundation.
[2] Materially supported by a NSERC operating grant.

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