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A GENERALIZED NONCOMMUTATIVE KOROVKIN THEOREM AND $*$-CLOSEDNESS OF CERTAIN SETS OF CONVERGENCE

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Introduction

Let A be a complex C^*-algebra with identity 1_A, and for $n = 1, 2, \ldots$, let $\phi_n : A \to A$ be a Schwarz map, i.e., a *linear map such that

$$\phi_n(a)^* \phi_n(a) \leq \phi_n(a^*a)$$

for all $a \in A$. Robertson [4] has proved that the set

$$C = \{a \in A : \|\phi_n(a) - a\| \to 0, \|\phi_n(a^*a) - a^*a\| \to 0, \|\phi_n(aa^*) - aa^*\| \to 0\}$$

is a C^*-subalgebra of A. This is a noncommutative analogue of a classical theorem of Korovkin which states that if $A = C([a, b])$, the set of all continuous functions on $[a, b]$, and $\phi_n : A \to A$ is a positive map for $n = 1, 2, \ldots,$ then

$$C = \{f \in A : \phi_n(f) \to f, \phi_n(|f|^2) \to |f|^2 \text{ uniformly on } [a, b]\}$$

is a norm-closed and conjugate closed subalgebra of A; in particular, if, 1, t and t^2 belong to C, then by the Stone-Weierstrass theorem, $C = C([a, b])$.

Let B be another C^*-algebra with identity 1_B, $\phi : A \to B$ a *homomorphism, and, for $n = 1, 2, \ldots$, $\phi_n : A \to B$ a Schwarz map. Note that each ϕ_n is a positive map with $\phi_n(1_A) \leq 1_B$. Consider the set

$$D = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^*a) \to \phi(a^*a)\},$$

where the convergence is in the norm topology or in the weak topology. In Section 1, we show that the set D is a norm-closed (but not necessarily $*$-closed) subalgebra of A (Theorem 1.2). By considering $D \cap D^*$, we obtain a straightforward generalization of Robertson's result (Corollary 1.4).

In case A is commutative, the set D is clearly $*$-closed. The purpose of this paper is to investigate the $*$-closedness of the set D in case A is noncommutative. Let $B = A$ and ϕ be the identity map. Robertson has asked whether the $*$-closedness of the set D for all choices of Schwarz maps ϕ_n characterizes the commutativity of A. We answer this question in the negative by using the theorem proved in Section 1. We show that if $A = M_2$, the noncommutative C^*-algebra consisting of all 2×2 complex matrices, then the set D is $*$-closed

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