EQUIVALENCE CLASSES OF PERIODIC SEQUENCES[1]

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1. Introduction

Fine [1], in 1958, and Gilbert and Riordan [2], in 1961, treated the following "necklace" problem [3]: if two periodic sequences a and a' can be made alike either by a shift in origin or a permutation of states, or both, how many inequivalent symmetry types of sequences are there? A finite-state sequential machine capable of generating a sequence a is capable of also generating any sequence a' equivalent to a, since shifts in the origin can be set by proper initial conditions fed into the machine, and since permutations of the states are mere relabelings of the output states. The solution to the necklace problem is then the number of finite-state machines required to generate all sequences having a given period.

If f is a real or complex function on the sequence objects, the autocorrelation of a relative to f is defined as

$$R_f(m) = (1/p) \sum_{n=1}^{p} f(a_n) f^*(a_{n+m}),$$

where p is the period of a. This function f represents, for example, the way sequence states are interpreted as signal voltages for transmission through a communications channel, and the values of $R_f(m)$ are related to the probability that an optimum receiver makes an error when attempting to determine the origin of a in the presence of noise [4].

By considering periodic sequences generated under the mapping

$$a_n \longrightarrow a_{kn+t},$$

one can show, when $(k, p) = 1$, that

$$R_f(m) \longrightarrow R_f(km).$$

Such a mapping is thus important in coding theory, because the values assumed by the autocorrelation function of a are left invariant, signifying that these sequences have the same correlation properties.

If a and a' are called *equivalent* whenever there exist t and k, $(k, p) = 1$, such that $a_n = a'_{kn+t}$, then all sequences in an equivalence class have the same correlation values. By application of Pólya's lemma [5] to all such transformations of the type $a_n \longrightarrow a_{nk+t}$, it is possible to count the number of these equivalence classes.

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